The Generalized Method of Phase Trajectory Curvature Synthesis in Spacecraft Attitude Dynamics Tasks

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The attitude dynamics of a variable mass spacecraft is considered. The variability of mass-inertia parameters in time is associated with operation of a jet-engine, which forms an interorbital impulse. The thrust of the jet-engine is directed along the longitudinal spacecraft axis, which fulfills the nutational-precessional motion relative to a planned direction in the inertial space. Therefore, to increase an accuracy of the thrust impulse formation during the jet-rocket engine work, and to reduce dispersion of the thrust due to the precessional rotation, it is needed to synthesize the motion regime with monotonously decreasing nutation angle. Such regimes allow to obtain a spiral twisting of the longitudinal axis to the predefined target direction, when the thrust will be focusing to target direction. To analyze and to synthesize this nutational-precessional motion properties, the curvature of the corresponded phase trajectories is evaluated in an angular phase space. The phase trajectory curvature evaluation gives an opportunity to synthesize the appropriate time-dependencies of inertia-mass parameters of spacecraft, which allows to increase the accuracy of the thrust impulse formation during operation of the jet-rocket engine. The proposed appropriate technique for positive phase trajectory synthesis is based on the previously developed method of curvature evaluation.

Keywords: spacecraft, variable mass, angular momentum, phase trajectory curvature

1. Introduction

In this paper, the attitude dynamics of variable mass spacecraft (SC) is analyzed and synthesized during the operation of solid propellant jet-engines. This theme belongs to a very reach scientific area of the angular motion dynamics of a rigid body, which includes many dynamical aspects associated with important fundamental results and practical applications in the framework of space flight mechanics of spacecraft and satellites. The many different tasks of the satellites motion stability analysis were considered, for example, in [1–13]. The important regimes of permanent rotations, regular precessions, and related aspects, including the modified classical cases of rigid bodies dynamics with additional perturbations were studied in [2,10,13–17].

Separately, we should indicate classical and modern questions and results in the field of variable mass bodies dynamics, which are collected and discussed in [18–32]. First of all, among these results, the main mathematical model of the attitude motion of variable mass bodies is constructed on the base of classical works [19–21,23] and on results obtained in [25–27], which include the qualitative method of the angular motion regimes synthesis. Despite the wide range of formulations of the solved problems, it is worth to note that the problem of the variable mass systems is still relevant for considerations in new statements [33,34]. And, moreover, different types of the change of the mass-inertia parameters are investigated in the framework of a spacecraft constructions transformations [32], or at the selection of different laws of jet-engines thrust [28].

In the development of the theme of variable structure systems motion, it is important to note in advance, that as the result of this paper will be related to the conditions of formulation allowing realization of the precessional motion modes with natural decreasing the nutation — such modes are advantageous and preferred in some cases of spacecraft attitude dynamics. E.g., these conditions are useful for practical applications in the area of the space missions with interorbital maneuvers, where the formation of the interorbital impulse can be passively refined in its direction during engine operation only due to the inertia-mass parameters changing at which the decreasing of the nutation angle is fulfilling (we need to synthesize and to implement such laws of change in the moments of inertia, which provide the natural passive decreasing of the nutation without any external and reactive torques). Such missions use the simplest type of the gyroscopic attitude stabilization with the help of spin-up of the spacecraft around its own axis. The spacecraft has the precessional motion with passively focusing the direction of the engine thrust, which is directed along the rotation axis.

To underline the novelty and the importance of the results obtained, we also can remind for comparison the well-known effect of damping of transverse (nutational) vibrations by short jet-force-impulses from orthogonal jet-engines, which decrease the nutation in an active way [22]. In [22] a possible alignment of the gyroscope axes is described in cases of the constant mass gyroscope without any consideration of the mass changing at the jet-engines operation. So, the dynamical task of the focusing and correcting gyroscope axis presented in this research is known and still relevant, especially if it can be realized by the passive way due to the right choice of the laws of the mass-inertia parameters changing.

2. The mathematical model of the angular motion of variable mass rigid bodies

Let us write the main dynamical equations with the help of the angular momentum change theorem. We can use the following coordinate systems (fig. 1).

The frame $OXYZ$ is a moving coordinates system with axes that are parallel to the inertial axes $P\xi\eta\zeta$. The point O coincides with the initial position of the center of mass of the system at the initial time-moment.

In the moving axes $OXYZ$ the vector equation can be written in the form, obtained in the classical treatise [20]:

$$\frac{d\vec{K}_0}{dt} = \vec{M}_0^e + \vec{M}_0^r + \sum_{\nu=1}^{n} \vec{\rho}_\nu \times \frac{dm_\nu}{dt}\left(\vec{\omega}\times\vec{\rho}_\nu\right) - \vec{\rho}_C \times M\vec{w}_0, \tag{1}$$

where $\vec{K}_0$ — the angular momentum vector; $\vec{M}_0^e$ — the external torques vector; $\vec{M}_0^r$ — the reactive jet-forces torques vector; $\vec{\rho}_\nu\left(x_\nu, y_\nu, z_\nu\right)$ — the radius vector of the body point with the index ν; $\vec{\rho}_C = \vec{\rho}_C\left(t\right)$ — the center of mass radius vector at the current time; $m_\nu = m_\nu\left(t\right)$ — the mass of the body point with index ν; $\vec{\omega} = \left(p, q, r\right)$ — the angular velocity vector; $M = M\left(t\right) = \sum_\nu m_\nu\left(t\right)$ — the body mass at the current time; $\vec{w}_0$ — the acceleration of the point O which moves in the inertial space (this point is selected per pole).

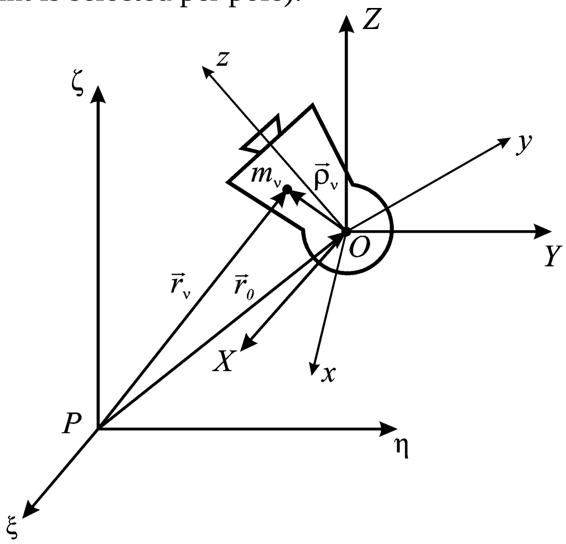

Fig. 1. Coordinate systems

Here we must note, that the equation (1) represents the fundamental result, obtained in the framework of the mechanics of the body of variable mass by Arkadiy Alexandrovich Kosmodem'ianskii [20] on the base of the well known classical theory of a variable mass point dynamics by Ivan Vsevolodovich Meshcherskiy [19]. All subsequent mathematical deductions are fullfiled on the grounds of the equation (1), which defines the law ("the theorem") of the changing an angular momentum of the system of variable mass relative to a moving coordinates frame, which axes are always parallel to axes of an inertial coordinates system.

Let us introduce the axes system $Oxyz$, that is invariably connected to the moving body and choose the point O, in which the center of mass is located at the initial moment of time, as the reference pole. Then

$$\frac{d\vec{K}_0}{dt} = \frac{\tilde{d}\vec{K}_0}{dt} + \vec{\omega}\times\vec{K}_0, \tag{2}$$

where $\tilde{d}/dt$ is the local derivative with respect to the $Oxyz$ axes.

Substituting (2) into (1) gives

$$\frac{\tilde{d}\vec{K}_0}{dt}+\vec{\omega}\times\vec{K}_0=\vec{M}_0^e+\vec{M}_0^r+ \\ +\sum_{\nu=1}^{n}\vec{\rho}_\nu\times\frac{dm_\nu}{dt}\left(\vec{\omega}\times\vec{\rho}_\nu\right)-\vec{\rho}_C\times M\vec{w}_0. \tag{3}$$

Let us rewrite (3) in a more convenient form in projections on axes *Oxyz* (further we omit the designation "~" in the notation $\tilde{d}/dt$):

$$\left[\frac{d\vec{K}_0}{dt}-\sum_{\nu=1}^{n}\vec{\rho}_\nu\times\frac{dm_\nu}{dt}\left(\vec{\omega}\times\vec{\rho}_\nu\right)\right]+\vec{\omega}\times\vec{K}_0= \\ =\vec{M}_0^e+\vec{M}_0^r-\vec{\rho}_C\times M\vec{w}_0. \tag{4}$$

In the body-fixed frame *Oxyz* we have the following form of the angular momentum and the center of mass vector:

$$\vec{K}_0=\hat{I}_0\cdot\vec{\omega};\quad \vec{\rho}_C=\left(x_C,\,y_C,\,z_C\right)^T, \tag{5}$$

where $\hat{I}_0$ is the tensor of inertia relative to the point *O*.

The local derivative takes the shape (the symbol "dot" indicates a time derivative):

$$\frac{d\vec{K}_0}{dt}=\frac{d}{dt}\left(\hat{I}_0\cdot\vec{\omega}\right)=\dot{\hat{I}}_0\cdot\vec{\omega}+\hat{I}_0\cdot\dot{\vec{\omega}}. \tag{6}$$

Let us write down the double vector product $\vec{\rho}_\nu\times\frac{dm_\nu}{dt}\left(\vec{\omega}\times\vec{\rho}_\nu\right)$ from (4):

$$\vec{\rho}_\nu\times\frac{dm_\nu}{dt}\left(\vec{\omega}\times\vec{\rho}_\nu\right)= \\ =\dot{m}_\nu\vec{\omega}\rho_\nu^2-\dot{m}_\nu\vec{\rho}_\nu\left(\vec{\omega}\cdot\vec{\rho}_\nu\right). \tag{7}$$

The scalar components of the vector result (7) are as follows:

$$\frac{dm_\nu}{dt}\vec{\omega}\rho_\nu^2=\begin{pmatrix}\dot{m}_\nu p\left(x_\nu^2+y_\nu^2+z_\nu^2\right)\\ \dot{m}_\nu q\left(x_\nu^2+y_\nu^2+z_\nu^2\right)\\ \dot{m}_\nu r\left(x_\nu^2+y_\nu^2+z_\nu^2\right)\end{pmatrix}; \tag{8}$$

$$\frac{dm_\nu}{dt}\vec{\rho}_\nu\left(\vec{\omega}\cdot\vec{\rho}_\nu\right)= \\ =\begin{pmatrix}\dot{m}_\nu\left(px_\nu^2+qx_\nu y_\nu+rx_\nu z_\nu\right)\\ \dot{m}_\nu\left(px_\nu y_\nu+qy_\nu^2+rz_\nu y_\nu\right)\\ \dot{m}_\nu\left(px_\nu z_\nu+qy_\nu z_\nu+rz_\nu^2\right)\end{pmatrix}. \tag{9}$$

Finally, the vector (7) can be written as follows:

$$\frac{dm_\nu}{dt}\vec{\omega}\rho_\nu^2-\frac{dm_\nu}{dt}\vec{\rho}_\nu\left(\vec{\omega}\cdot\vec{\rho}_\nu\right)= \\ =\dot{m}_\nu\left[\begin{pmatrix}\left(y_\nu^2+z_\nu^2\right)p\\ \left(x_\nu^2+z_\nu^2\right)q\\ \left(x_\nu^2+y_\nu^2\right)r\end{pmatrix}-\begin{pmatrix}x_\nu y_\nu q\\ x_\nu y_\nu p\\ x_\nu z_\nu p\end{pmatrix}-\begin{pmatrix}x_\nu z_\nu r\\ z_\nu y_\nu r\\ y_\nu z_\nu q\end{pmatrix}\right]. \tag{10}$$

According to its definition, the inertia tensor component I_{XX} corresponds to the value:

$$I_{XX}=\sum_{\nu=1}^{n}m_\nu\left(y_\nu^2+z_\nu^2\right). \tag{11}$$

After differentiating we have:

$$\dot{I}_{XX}=\frac{d}{dt}\sum_{\nu=1}^{n}m_\nu\left(y_\nu^2+z_\nu^2\right)= \\ =\sum_{\nu=1}^{n}\frac{dm_\nu}{dt}\left(y_\nu^2+z_\nu^2\right)+\sum_{\nu=1}^{n}m_\nu\frac{d}{dt}\left(y_\nu^2+z_\nu^2\right). \tag{12}$$

Since the body is absolutely rigid $\left(\forall\,\nu:\quad x_\nu=\mathrm{const},y_\nu=\mathrm{const},\ z_\nu=\mathrm{const}\right)$, then $\frac{d}{dt}\left(y_\nu^2+z_\nu^2\right)=0$ and the expression (12) takes the form:

$$\dot{I}_{XX}=\sum_{\nu=1}^{n}\frac{dm_\nu}{dt}\left(y_\nu^2+z_\nu^2\right) \tag{13}$$

and similar reasoning is done for all the inertia tensor components (I_{YY}, I_{ZZ}, I_{XY}, I_{XZ}, I_{YZ}).

Taking into account (11), we rewrite (10):

$$\frac{dm_\nu}{dt}\vec{\omega}\rho_\nu^2-\frac{dm_\nu}{dt}\vec{\rho}_\nu\left(\vec{\omega}\cdot\vec{\rho}_\nu\right)= \\ =\begin{pmatrix}\dot{I}_{XX} & -\dot{I}_{XY} & \dot{I}_{XZ}\\ -\dot{I}_{XY} & \dot{I}_{YY} & -\dot{I}_{YZ}\\ \dot{I}_{XZ} & -\dot{I}_{YZ} & \dot{I}_{ZZ}\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}=\dot{\hat{I}}_0\cdot\vec{\omega}. \tag{14}$$

Substituting (6) and (14) in the equation (4), we obtain:

$$\left[\dot{\hat{I}}_0\vec{\omega}+\hat{I}_0\dot{\vec{\omega}}-\dot{\hat{I}}_0\vec{\omega}\right]+\vec{\omega}\times\vec{K}_0= \\ =\vec{M}_0^e+\vec{M}_0^r-\vec{\rho}_C\times M\vec{w}_0$$

or

$$\hat{I}_0\dot{\vec{\omega}}+\vec{\omega}\times\vec{K}_0= \\ =\vec{M}_0^e+\vec{M}_0^r-\vec{\rho}_C\times M\vec{w}_0. \tag{15}$$

Now we should rewrite the vector $\vec{\rho}_C\times M\vec{w}_0$. To do this, we can write the theorem on the center of mass motion for a variable mass body:

$$M\vec{w}_c^{(e)}=\vec{R}^{(e)}+\vec{\Phi}_r, \tag{16}$$

where $\vec{R}^{(e)}$ — the external forces vector; $\vec{\Phi}_r$ — the reactive forces vector.

The relationship between the accelerations $\vec{w}_c^{(e)}$ and $\vec{w}_0$ is the following ($\vec{\varepsilon}=\dot{\vec{\omega}}$ is the angular acceleration):

$$\vec{w}_C^{(e)} = \vec{w}_0 + \vec{\varepsilon} \times \vec{\rho}_C + \vec{\omega} \times \vec{\omega} \times \vec{\rho}_C, \tag{17}$$

After substituting (17) into (16) we have:

$$M\left(\vec{w}_0 + \vec{\varepsilon} \times \vec{\rho}_C + \vec{\omega} \times \vec{\omega} \times \vec{\rho}_C\right) = \vec{R}^{(e)} + \vec{\Phi}_r. \tag{18}$$

From (18) we can write:

$$\begin{gathered} M\vec{w}_0 = \vec{R}^{(e)} + \vec{\Phi}_r - \\ -M\vec{\varepsilon} \times \vec{\rho}_C - M\vec{\omega} \times \vec{\omega} \times \vec{\rho}_C. \end{gathered} \tag{19}$$

Now the equation (15) takes the form:

$$\begin{gathered} \hat{I}_0 \dot{\vec{\omega}} + \vec{\omega} \times \vec{K}_0 = \vec{M}_0^e + \vec{M}_0^r - \\ -\vec{\rho}_C \times \left(\vec{R}^{(e)} + \vec{\Phi}_r - M\vec{\varepsilon} \times \vec{\rho}_C - M\vec{\omega} \times \vec{\omega} \times \vec{\rho}_C\right) \end{gathered}$$

or

$$\begin{gathered} \hat{I}_0 \dot{\vec{\omega}} + \vec{\omega} \times \vec{K}_0 = \\ = \vec{M}_0^e + \vec{M}_0^r - \vec{\rho}_C \times \vec{R}^{(e)} - \\ -\vec{\rho}_C \times \vec{\Phi}_r + M\vec{\rho}_C \times \vec{\varepsilon} \times \vec{\rho}_C + \\ +M\vec{\rho}_C \times \vec{\omega} \times \vec{\omega} \times \vec{\rho}_C. \end{gathered} \tag{20}$$

Let us rearrange the terms in (20):

$$\begin{gathered} \left(\hat{I}_0 \dot{\vec{\omega}} - M\vec{\rho}_C \times \dot{\vec{\omega}} \times \vec{\rho}_C\right) + \\ +\left(\vec{\omega} \times \vec{K}_0 - M\vec{\rho}_C \times \vec{\omega} \times \vec{\omega} \times \vec{\rho}_C\right) = \\ = \left(\vec{M}_0^e - \vec{\rho}_C \times \vec{R}^{(e)}\right) + \left(\vec{M}_0^r - \vec{\rho}_C \times \vec{\Phi}_r\right). \end{gathered} \tag{21}$$

Notice, that

$$\left(\vec{M}_0^e - \vec{\rho}_C \times \vec{R}^{(e)}\right) = \vec{M}_C^e, \tag{22}$$

$$\left(\vec{M}_0^r - \vec{\rho}_C \times \vec{\Phi}_r\right) = \vec{M}_C^r, \tag{23}$$

where C — the point at which the body center of mass is located at the current time.

Let us write componentwise the term $M\vec{\rho}_C \times \vec{\varepsilon} \times \vec{\rho}_C$, projecting them on the body axis coordinate system:

$$\begin{gathered} M\vec{\rho}_C \times \dot{\vec{\omega}} \times \vec{\rho}_C = \\ = M\left[\left(\rho_C^2 \delta_{ij} - \rho_{Ci}\rho_{Cj}\right)\right]_{i,j=1..3} \cdot \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \tilde{I}\dot{\vec{\omega}}, \end{gathered} \tag{24}$$

where

$$\tilde{I} = \begin{pmatrix} M\left(y_C^2 + z_C^2\right) & -Mx_C y_C & -Mx_C z_C \\ -Mx_C y_C & M\left(x_C^2 + z_C^2\right) & -My_C z_C \\ -Mx_C z_C & -My_C z_C & M\left(x_C^2 + y_C^2\right) \end{pmatrix},$$

$$\rho_{C1} = x_C,\ \rho_{C2} = y_C,\ \rho_{C3} = z_C,$$

and

$$\delta_{ij} = \begin{cases} 1, & if\ \ i = j; \\ 0, & if\ \ i \neq j. \end{cases}$$

$\tilde{I}$ is formally the inertia tensor (relative to the point O) of the mass-point C, which is the center of mass of the body at the current time.

Now we should rewrite the vector $M\vec{\rho}_C \times \vec{\omega} \times \vec{\omega} \times \vec{\rho}_C$ in projections on the body-fixed frame:

$$M\vec{\rho}_C \times \vec{\omega} \times \vec{\omega} \times \vec{\rho}_C = M\vec{\rho}_C \times \left[\vec{\omega}\left(\vec{\omega} \cdot \vec{\rho}_C\right)\right], \tag{25}$$

where

$$\begin{gathered} \vec{\omega}\left(\vec{\omega} \cdot \vec{\rho}_C\right) = \\ = \begin{pmatrix} p^2 x_C + pqy_C + prz_C \\ pqx_C + q^2 y_C + qrz_C \\ prx_C + qry_C + r^2 z_C \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}. \end{gathered} \tag{26}$$

The substituting (26) in (25) gives:

$$\begin{gathered} M\vec{\rho}_C \times \left[\vec{\omega}\left(\vec{\omega} \cdot \vec{\rho}_C\right)\right] = \\ -prx_C^2 - qrx_C y_C - r^2 x_C z_C\big) + \\ +\vec{k}\left(pqx_C^2 + q^2 x_C y_C + qrx_C z_C - \right. \\ \left. -p^2 x_C y_C - pqy_C^2 - pry_C z_C\right)\big]. \end{gathered} \tag{27}$$

We can note, that

$$\begin{gathered} Mqr\left(y_C^2 - z_C^2\right) = \\ = qr\left(M\left(x_C^2 + y_C^2\right) - M\left(x_C^2 + z_C^2\right)\right). \end{gathered} \tag{28}$$

Using this technique, we rewrite (27) in the form:

$$\begin{gathered} M\vec{\rho}_C \times \left[\vec{\omega}\left(\vec{\omega} \cdot \vec{\rho}_C\right)\right] = \\ = \vec{\omega} \times \left(\tilde{I} \cdot \vec{\omega}\right) = \vec{\omega} \times \vec{\tilde{K}}. \end{gathered} \tag{29}$$

where $\vec{\tilde{K}} = \tilde{I} \cdot \vec{\omega}$ is formally the angular momentum (relative to the point O) of the mass-point C (with the mass M), which is the body center of mass at the current time.

Taking into account (22), (23), (24), (29) we can rewrite the equation (21):

$$\begin{gathered} \left(\hat{I}_0 \dot{\vec{\omega}} - \tilde{I}\dot{\vec{\omega}}\right) + \left(\vec{\omega} \times \vec{K}_0 - \vec{\omega} \times \vec{\tilde{K}}\right) = \\ = \left(\vec{M}_0^e - \vec{\rho}_C \times \vec{R}^{(e)}\right) + \left(\vec{M}_0^r - \vec{\rho}_C \times \vec{\Phi}_r\right). \end{gathered} \tag{30}$$

Let us present the following tensor and the vector as:

$$\hat{I}_C(t) = \hat{I}_0 - \tilde{I}, \tag{31}$$

$$\vec{K}_C = \vec{K}_0 - \vec{\tilde{K}}. \tag{32}$$

Then, the equation (30) can be written in the form:

$$\hat{I}_C(t) \cdot \dot{\vec{\omega}} + \vec{\omega} \times \vec{K}_C = \vec{M}_C^e + \vec{M}_C^r. \tag{33}$$

The Equation (33) corresponds to the main law of the change of angular momentum of variable mass bodies.

Formally, the law (33) can be considered as the form of the angular momentum change theorem presented in the moving coordinate system with the origin in the center of mass C (at the current time) and with axes, which are parallel to the frame *Oxyz* connected to the body. Also we should note that the main pole *O* can move with any linear acceleration (since the center of mass *C* moves along the orbit), so the axes *OXYZ* are not inertial in the common case.

Here we should note that the rigid body center of mass with variable mass can change its own position relative to the rigid body due to change of masses of its different points. So, even if the body is stationary in inertial space, its center of mass can move due to a change of mass of its points (for example, when a stationary rod burns, the center of mass formally moves towards unburned points). In other words, the center of mass is not a fixed point in the composition of a rigid body of variable mass.

In this regard, we must know the specific laws of the inertia-mass geometry time-dependencies of a body of variable mass. Only then we can calculate all of the rigid body inertia-mass parameters, like the set of constants and time-functions, written for the rigid body in the body-fixed frame *Oxyz*:

$$set = \left\{ \hat{I}_0(t),\, \tilde{I}(t),\, x_C(t),\, y_C(t),\, z_C(t) \right\}. \quad (34)$$

So, if we know the set (34), then we can directly use the form of the dynamical equations (30) or the form (33). In any case, the vector equation (33) was built in the framework of the consideration of all vector values in the principal body coordinates *Oxyz*; the form (33) corresponds to the formal rewriting of the main equation (30), obtained in *Oxyz*.

The vector equation (33) demonstrates the main meaning of the law of the angular momentum change in the application to the consideration of the variable mass body: as we can see, the main sense is not differing from the case of the body of constant mass.

The presented process of the equation (30) (or (33)) obtaining corresponds to the ideology, which was used in classical works [20,23]. Such detailed deduction of the equation (30) is presented by this paper, due to the importance and the non-obviousness of the final form, where the derivatives from the components of the inertia tensor $\dot{\hat{I}}_0(t)$ (or $\dot{\hat{I}}_C(t)$) are absent.

It should be noted that in works [20,23] a similar result has already been obtained for a more simple case, when a variable mass body has the principal axes variable inertia tensor and always (at any following time-moment) remains its form.

Within present consideration the variable inertia tensor $\hat{I}_0$ can have any common form (in the presence of centrifugal moments of inertia); also the variable position of the center of mass relative to the body can be arbitrary and can arbitrary move inside the body (i.e. it can change its calculated coordinates relative to the body-fixed frame due to the mass variability and, therefore, with any complex form of $\tilde{I}$).

3. The main applied task formulation

Now, basing on the main equations (30) or (33) we can formulate the main conditions of our further research in the framework of applied problems of attitude dynamics of SC with variable inertia-mass parameters.

Let us consider the SC with an operating jet-engine, which forms an interorbital impulse in a "target direction", which coincides in our case with axis *OZ*.

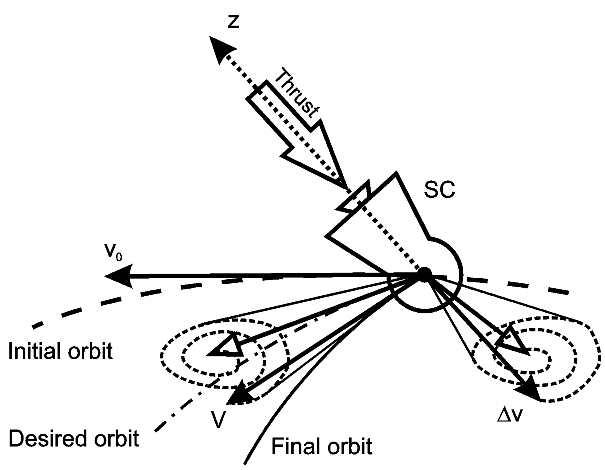

Fig. 2. SC with a jet-engine

The thrust of the operating jet-engine is directed along the longitudinal spacecraft axis *Oz*. Let us assume that the gyroscopic stabilization of the SC and its thrust vector is fulfilled by the fast rotation around the longitudinal axis *Oz* in the target direction *OZ*. In other words, we have the task stabilization of the SC precessional motion along targeted direction *OZ*, with changing inertia-mass parameters of the rigid body (SC).

Therefore, to increase an accuracy of the thrust impulse formation during the jet-rocket engine work, and to reduce dispersion of the thrust due to the precessional rotation, it is needed to synthesize the motion regime with monotonously decreasing nutation angle. Such regimes allow to obtain a spiral twisting of the longitudinal axis to the predefined target direction, when the thrust will be focusing to target direction.

Let us consider the case of the SC motion, when body-fixed frame remain principal all the time, then the inertia tensor will have the diagonal form:

$$\hat{I}_0 = diag\left(A_0(t), B_0(t), C_0(t)\right). \tag{35}$$

Let us assume, that the center of mass during the mass changing process always remains on the longitudinal axis *Oz* axis:

$$\vec{\rho}_C = \left(0, 0, z_C(t)\right)^T. \tag{36}$$

Then (24) and (25) take the form:

$$M\vec{\rho}_C \times \vec{\varepsilon} \times \vec{\rho}_C + M\vec{\rho}_C \times \vec{\omega} \times \vec{\omega} \times \vec{\rho}_C = = Mz_C^2 \begin{pmatrix} \dot{p} - qr \\ \dot{q} + pr \\ 0 \end{pmatrix}. \tag{37}$$

In this conditions, the main dynamical equations can be written as follows:

$$\begin{cases} A(t)\dot{p} + \left(C(t) - B(t)\right)qr = M_{Cx}^e + M_{Cx}^r, \\ B(t)\dot{q} + \left(A(t) - C(t)\right)pr = M_{Cy}^e + M_{Cy}^r, \\ C(t)\dot{r} + \left(B(t) - A(t)\right)pq = M_{Cz}^e + M_{Cz}^r, \end{cases} \tag{38}$$

where

$$\begin{cases} A(t) = A_0(t) - M(t)z_C^2(t); \\ B(t) = B_0(t) - M(t)z_C^2(t); \\ C(t) = C_0(t). \end{cases}$$

In addition, in our research we also will use the following assumptions.

Assumption A1. The SC motion is considered on a finite time interval, where the values of decreasing physical quantities of masses and inertia moments remain always positive, that is, the problem physical statement correctness ensuring.

Assumption A2. The spacecraft fulfils the angular motion with small values of the equatorial components of the angular velocity in comparison with the longitudinal component (that is usually realized in cases of the gyroscopic stabilization of the jet-engine thrust vector direction):

$$\frac{\sqrt{p^2 + q^2}}{r} = \varepsilon \ll 1 \tag{39}$$

To describe the attitude motion we will use the Euler-type angles, corresponding to three rotations in the following order:

$$\psi(\text{around } x) \to \gamma(\text{around } y) \to \varphi(\text{around } z).$$

Such angles also called as the Krylov's angles.

The kinematical equations system has the form:

$$\begin{aligned} \dot{\gamma} &= p\sin\varphi + q\cos\varphi, \\ \dot{\psi} &= \frac{1}{\cos\gamma}\left(p\cos\varphi - q\sin\varphi\right), \\ \dot{\varphi} &= r - \frac{\sin\gamma}{\cos\gamma}\left(p\cos\varphi - q\sin\varphi\right). \end{aligned} \tag{40}$$

So, further we will investigate the SC angular motion basing on the dynamical equations (38), kinematical equations (40) considering the *assumptions A1* and *A2*.

4. The generalization of the method of phase trajectories forms analysis by their curvature evaluation

To analyze the spacecraft dynamics, we will apply the "phase trajectory curvature method", obtained and well-tried in works [26,27]. Here in this paper we consider the developing of this method to investigate the overaged forms of phase trajectories of the variable mass spacecraft angular dynamics in the cases of dynamical asymmetry $\left(A(t) \neq B(t)\right)$.

As it was described earlier, to analyze the dynamics of a non-autonomous system the special method is very useful [26,27], which allows to evaluate the qualitative properties of phase trajectories, and even to synthesize their forms.

According to this method [26,27] let us change the variables for describing the equatorial components of the angular velocity of SC:

$$\begin{aligned} p &= G(t)\sin F(t), \\ q &= G(t)\cos F(t). \end{aligned} \tag{41}$$

The assumption 3 allows us to consider the angles γ and ψ as small values $\left(\gamma = O(\varepsilon),\right.$ $\left.\psi = O(\varepsilon)\right)$. Taking into account (41) and assumption *A2*, equations (40) take the form:

$$\dot{\gamma} = G\cos\Phi(t),\ \dot{\psi} = G\sin\Phi(t),\ \dot{\varphi} = r, \tag{42}$$

where

$$\Phi(t) = F(t) - \varphi(t). \tag{43}$$

After substituting (41) in equations (38) we get:

$$
\begin{cases}
\dot{F} = -\dfrac{r}{AB}\Big[B(C-B)\cos^2 F + \\
+A(C-A)\sin^2 F\Big] + \\
+\dfrac{M_x}{A}\cos F - \dfrac{M_y}{B}\sin F, \\
\dot{G} = -G\dfrac{r}{AB}\sin F\cos F \times \\
\times (B-A)(C-A-B) + \\
+\dfrac{M_x}{A}\sin F + \dfrac{M_y}{B}\cos F, \\
\dot{r} = -G^2\dfrac{B-A}{C}\sin F\cos F + \dfrac{M_z}{C},
\end{cases} \tag{44}
$$

Performing some algebraic transformations in (44), we get:

$$
\begin{cases}
\dot{F} = \dfrac{1}{2}r\left(\alpha - \beta\cos 2F\right) + \eta_1, \\
\dot{G} = \dfrac{1}{2}Gr\beta\sin 2F + \eta_2, \\
\dot{r} = \dfrac{1}{2}G^2\xi\sin 2F + \eta_3,
\end{cases} \tag{45}
$$

where

$$
\xi_1 = \frac{B}{A} - \frac{C}{A}, \quad \xi_2 = \frac{A}{B} - \frac{C}{B}, \quad \xi_3 = \frac{A}{C} - \frac{B}{C};
$$
$$
M_x = M^e_{Cx} + M^r_{Cx}, \; M_y = M^e_{Cy} + M^r_{Cy},
$$
$$
M_z = M^e_{Cz} + M^r_{Cz};
$$
$$
\alpha = \xi_1 + \xi_2, \; \beta = \xi_1 - \xi_2; \tag{46}
$$
$$
\eta_1 = \frac{M_x}{A}\cos F - \frac{M_y}{B}\sin F,
$$
$$
\eta_2 = \frac{M_x}{A}\sin F + \frac{M_y}{B}\cos F, \quad \eta_3 = \frac{M_z}{C}.
$$

From spherical geometry the formula for a nutation angel θ (the angle between axes OZ and Oz) follow $\cos\theta = \cos\psi\cos\gamma$. We consider the small angles γ and ψ, and then the nutation angle can be defined by the following approximated formula:

$$
\theta^2 \cong \gamma^2 + \psi^2. \tag{47}
$$

So, taking into account the last comment, the phase plane $\{\gamma, \psi\}$ can be considered as the phase space to effective describing the nutational-precessional motion. The phase trajectory in this space completely characterizes motion of the longitudinal axis Oz (of an apex of the longitudinal axis).

Therefore, our further research will be dedicated to the analysis of this planar phase space $\{\gamma, \psi\}$, and the main question here will be linked to the shapes of phase trajectories on this plane.

On the indicated plane, the phase point will formally have its components of a phase velocity and a phase acceleration:

$$
V_\gamma = \dot{\gamma}, \; V_\psi = \dot{\psi}, \; W_\gamma = \ddot{\gamma}, \; W_\psi = \ddot{\psi}.
$$

Then with the help of expressions (42) the curvature of a phase trajectory (k) is evaluated as follows:

$$
k^2 = \left(\ddot{\gamma}\dot{\psi} - \ddot{\psi}\dot{\gamma}\right)^2 \Big/ \left(\dot{\gamma}^2 + \dot{\psi}^2\right)^3 = \dot{\Phi}^2 / G^2. \tag{48}
$$

If the curvature magnitude increases, there will be a motion with a twisted spiral trajectory similar to a steady focal point (Fig. 5, case “a”) and if decreases — with an untwisted spiral. Therefore, to obtain the twisted spiral trajectory, we need to have the monotonously increasing curvature value, i.e.:

$$
|k| \uparrow \Rightarrow k\dot{k} > 0 \;\Rightarrow\; \dot{\Phi}\ddot{\Phi}G - \dot{G}\dot{\Phi}^2 > 0. \tag{49}
$$

Therefore, to analyze the form of a phase trajectory on the plane $\{\gamma, \psi\}$, and to evaluate the condition (49) fulfilling, it is necessary to study a disposition of roots of the following function, which we will call as “the curvature function”:

$$
P(t) = \dot{\Phi}\ddot{\Phi}G - \dot{G}\dot{\Phi}^2. \tag{50}
$$

The function (50) can be called as the “function of phase trajectories evolutions”.

Different qualitative cases of behaviors of phase trajectories are possible depending on function $P(t)$. In the first case (Fig. 3, case “a”) the function is always positive and has no zeros on whole considered interval of time $t \in [0, T]$, and the phase trajectory is a twisting spiral. If the function $P(t)$ has one zero root (Fig. 5, case “b”), then one change of quality of the dynamics takes place, and as the result we obtain the the Cornu’s spiral (also known as the “clothoid”) case “b”. If $P(t)$ has many roots, then the phase trajectory has alternation of untwisted and twisted segments (Fig. 5, case “c”, “d”).

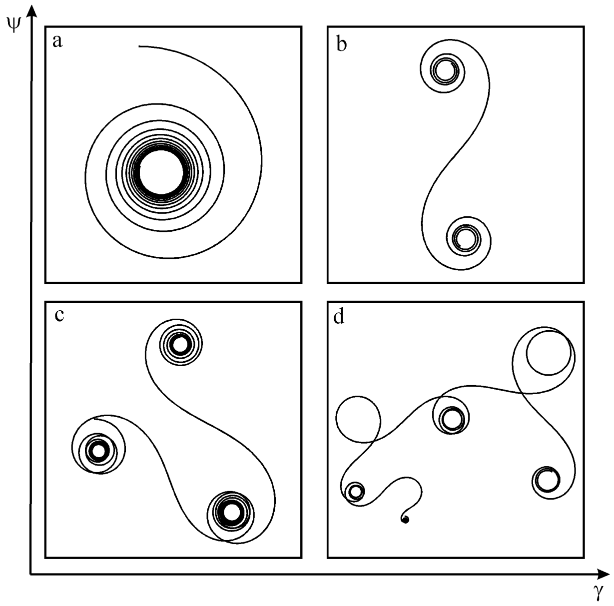

Fig. 3 The hodographs of the longitudinal SC axis (*Oz*) on the tangential plane *{ψ-γ}*

Returning to the task of the spacecraft attitude dynamics synthesis and as we already sad, the variability of mass-inertia parameters in time related to the operation of a jet-engine, which forms an interorbital impulse. The thrust of the jet-engine is directed along the longitudinal spacecraft axis *Oz*, which fulfills the nutational-precessional motion relative to a planned direction in the inertial space *OZ*. Therefore, to increase an accuracy of the thrust impulse formation during the jet-rocket engine work, and to reduce dispersion of the thrust due to the precessional rotation, it is needed to synthesize the motion regime with monotonously decreasing nutation angle. Such regimes correspond to the spiral twisting of the longitudinal axis to the predefined target direction, when the thrust will be focusing to target direction. So, to obtain this spiral twisting dynamics we need to analyze the function (50), and to guarantee/synthesize its always positive value $P(t)>0$ (without roots on the whole time of the jet-engine operation) — this task is equivalent to the synthesis of the time-dependent mass-inertia parameters, which define the properties of the fuel internal allocation, the shapes of tanks and the laws of masses change in time.

If we rewrite (50) then we obtain:

$$P = P\left(t, F, G, r\right) = \\ = \frac{\dot{F}-r}{G}\cdot\frac{d}{dt}\left(\frac{\dot{F}-r}{G}\right) \tag{51}$$

As we have already mentioned above, to achieve the desired dynamics, the function *P* must be positive over the entire considered time-interval.

The substitution of (45) into (51) gives the following explicit expression for the function *P*:

$$\begin{gathered} P = \frac{r\left(\alpha-2-\beta\cos 2F\right)+2\eta_1}{4G^2}\times \\ \times\Big[\eta_3\left(\alpha-2\right)+r\dot{\alpha}+ \\ +\left(\frac{1}{2}\xi G^2-2\beta r^2-\frac{2\eta_2 r}{G}\right)\times \\ \times\left(\alpha-\beta\cos 2F\right)\sin 2F+ \\ +\left(2\beta r^2-\xi G^2-4\beta\eta_1 r\right)\sin 2F+ \\ +\left(\dot{\beta}r-\beta\eta_3\right)\cos 2F+\frac{4}{G}\left(r-\eta_1\right)\eta_2\Big]. \end{gathered} \tag{52}$$

The form of evolution function (52) is difficult to analyze its analytical structure and to synthesize the positive inertia-mass parameters due to presence of oscillating terms with quite large frequencies. If we average the function *P* on the period of *F*, we can get more convenient form:

$$\left\langle P\right\rangle = \frac{r}{4G^2}\left(\alpha-2\right)\left(\frac{M_z}{C}\left(\alpha-2\right)+r\dot{\alpha}\right), \tag{53}$$

where

$$\begin{gathered} \alpha = \frac{A}{B}+\frac{B}{A}-\frac{C}{A}-\frac{C}{B}, \\ \dot{\alpha} = \left[\frac{\dot{A}}{B}+\frac{\dot{B}}{A}-\frac{\dot{C}}{A}-\frac{\dot{C}}{B}\right]- \\ -\left[A\frac{\dot{B}}{B^2}+B\frac{\dot{A}}{A^2}-C\frac{\dot{A}}{A^2}-C\frac{\dot{B}}{B^2}\right]. \end{gathered} \tag{54}$$

Here the designation $\langle\bullet\rangle$ means the averaging on *F*-period:

$$\langle\bullet\rangle = \int\limits_0^{2\pi}\left(\bullet\right)dF.$$

Basing on the expression (53), we can further investigate the qualitative properties of the different cases of variable mass spacecraft attitude dynamics in its averaged form.

5. The synthesis of the attitude dynamics

Let us consider some cases of the spacecraft dynamics at different time-dependencies of inertia-mass parameters.

Case 1. The dynamic symmetry of SC.

First of all, we should consider the dynamics of the dynamically symmetrical spacecraft. In this case we always have equality of the equatorial inertia moments $A(t)\equiv B(t)>0$. Then the following values are actual:

$$\alpha = 2\left(1-\frac{C}{A}\right), \quad \dot{\alpha} = -2\frac{d}{dt}\left(\frac{C}{A}\right). \tag{55}$$

$$\langle P\rangle = \frac{rC}{G^2A^2}\left(M_z + \frac{r}{A}\left(\dot{C}A - \dot{A}C\right)\right). \tag{56}$$

The condition of positive values of the evolution function in this case is:

$$rM_z + \frac{r^2}{A}\left(\dot{C}A - \dot{A}C\right) > 0. \tag{57}$$

So, if the condition will be fulfilled, then the spacecraft will have the positive dynamics with twisted spiral for the apex of the longitudinal axis and for the vector of thrust. The considered case has the analytical description and it is fully consistent with the previous results [26,27].

Case 2. The absence of external and reactive torques.

Now we consider the case of the angular motion of the SC with three-axial general inertia tensor (the dynamically asymmetrical rigid body). If there are no external and reactive moments in (53), then

$$\begin{aligned}
\langle P\rangle &= \frac{r}{4G^2}(\alpha-2)(0+r\dot{\alpha}) = \\
&= \frac{r}{4G^2}(\alpha-2)\left(0+r\frac{d}{dt}(\alpha-2)\right) = \\
&= \frac{r^2}{4G^2}(\alpha-2)\frac{d}{dt}(\alpha-2) = \\
&= \frac{r^2}{2G^2}\frac{d}{dt}(\alpha-2)^2.
\end{aligned} \tag{58}$$

The condition $\langle P\rangle > 0$ will have the simple analytical form:

$$\frac{d}{dt}(\alpha-2)^2 > 0. \tag{59}$$

So, for the positive dynamics it is enough to take such inertia-mass parameters, that correspond to the condition (59) — such parameters will guarantee the positive angular motion of SC, when the thrust vector (along *Oz*) spirally focuses along the target direction *OZ* by the natural way.

6. Numerical Simulation results

In the previous section, the analytical conditions were obtained for some important special cases, but in arbitrary common case, we cannot find the analytical evaluations. Then we need to carry out a direct numerical calculation of the evolution function form, to make sure that the parameters are positive (if it is confirmed that $P(t)>0$), or otherwise.

Below comparative calculations for the full (52) and averaged (53) shapes of the evolution function are presented (blue lines correspond to the full shape, red lines — to averaged shape). These results show the fulfilment of our main idea: firstly, the full and the averaged results are close to each other, and, secondly, the averaged shape of the evolution function quite well defined the main tendencies of the phase trajectory ("twisting", "unwinding" and “alternating” evolutions).

Equation (53) shows, that the motion quality will change in cases when the multipliers signs are altering:

$$\operatorname{sign}(\alpha-2) = \pm 1 \quad \text{and/or}$$

$$\operatorname{sign}\left(\frac{M_z}{C}(\alpha-2) + r\dot{\alpha}\right) = \pm 1.$$

Therefore, the complex cases of evolutions can be fulfilled due to final signs alternations.

Let us provide the numerical modeling to show all features of the suggested method.

Example 1.

$$A(t) = 14 - 0.7t, \; B(t) = 13 - 0.65t,$$

$$C(t) = 5 - 0.1t, \; M_x = M_y = M_z = 0.$$

In this case, the inertia moments satisfy the simple condition $A(t) > B(t) > C(t)$ at $\forall t \in [0, T]$. Nevertheless, the evolution function $P(t)$ will be oscillating (fig.5, the blue line), that does not allow to understand the quality of the dynamics and to guarantee any predefined trajectories form. At the same time, the averaged form $\langle P\rangle$ has the monotonously growing positive trend (fig.5, the red line), so we can predict the spiral twisting form of the corresponding averaged phase trajectory (fig.4, the red line).

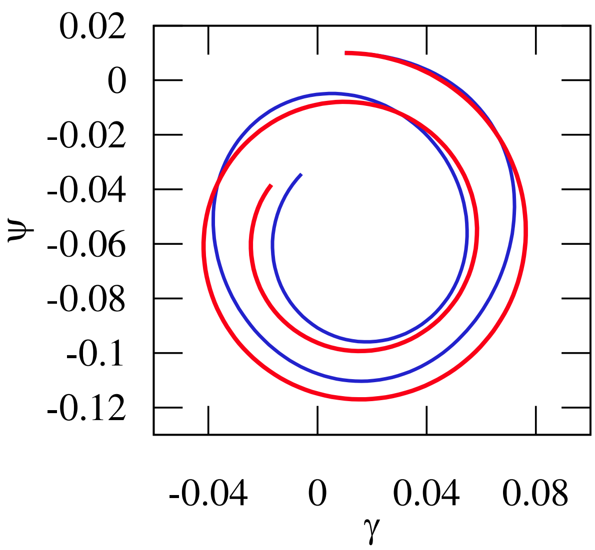

Fig. 4 The thrust vector hodograph evolution of Example 1

If we plot the phase trajectory without averaging, then we obtain its “direct” form (fig.4, the blue line), which corresponds to the general shape of the evolution function (52). It possible to conclude, that the direct form and the averaged form are close to each other. This similarity demonstrates the adequate applicability of the averaged evolution function (53) to qualitative analyzing of the phase trajectory form.

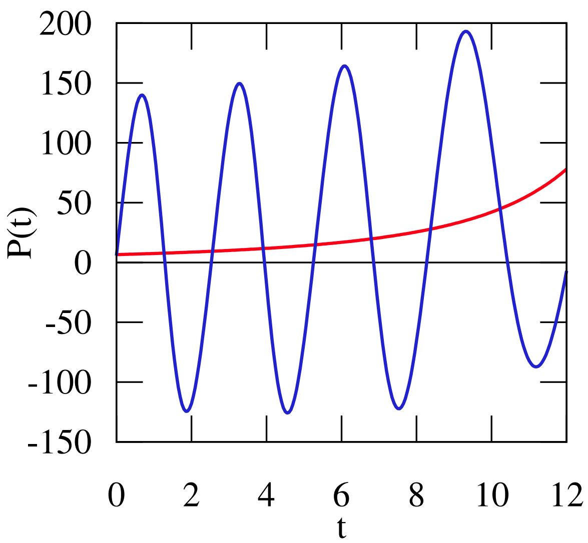

Fig. 5 The curvature function *P(t)* of Example 2

In addition, it is quite important to show the confirmation of our analytical predictions about the nutation angle behavior. The fig.6 contains the time-dependence of the nutation oscillation with decreasing amplitude (the blue line), and the averaged value of nutation (the red line). This behavior fully confirms the adequate effectiveness of the suggested method.

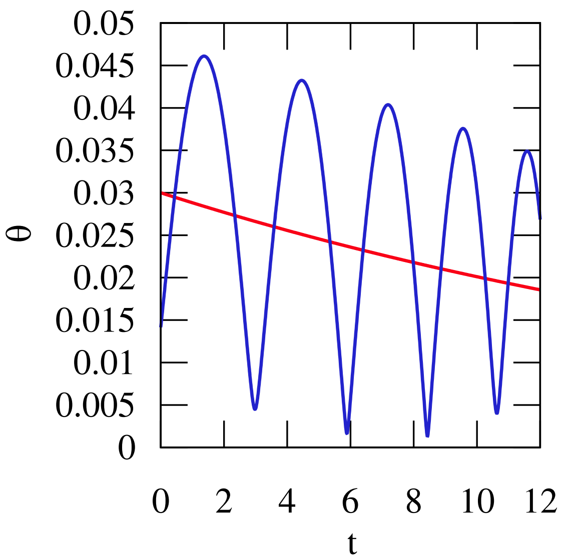

Fig.6 The nutation angle

Let us consider the next example, where we will take into account the constant external torque.

Example 2.

$$M_x = M_y = 0, \quad M_z = -2C(0).$$

$$A(t) = 40000 - 2000t,$$
$$B(t) = 44000 - 2200t, \qquad (60)$$
$$C(t) = 56000 - 2800t,$$

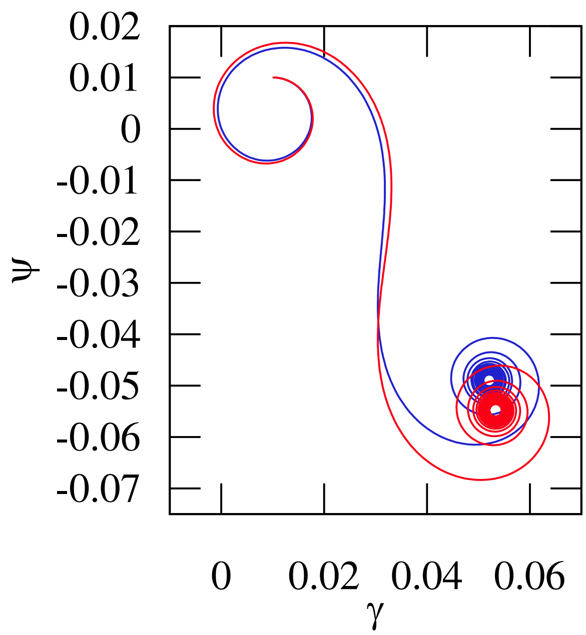

Fig. 7 The thrust vector hodograph evolution of Example 2

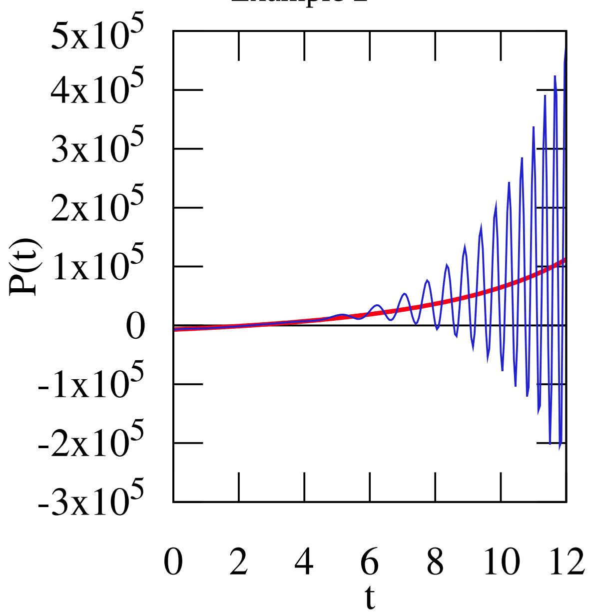

Fig. 8 The curvature function *P(t)* of Example 2

The numerical results (fig. 7, 8) as it was menitoned previously demonstrate the quite correspondence of direct (the blue line) and averaged (the red line) forms of phase trajectories (fig. 7). The direct and averaged forms of the evolution function are depicted at the fig. 8, which additionally confirms the expediency of using the averaged form. It is needed to note, that in comparison with the direct form of the phase trajectory, the averaged trajectory has some poor final position of the final twisting end — this, of course, means the accumulation of some numerical errors within the framework of using the method, but it does not affect the qualitative predictions of behavior in any way. We have, as the result, the adequate correspondence of direct and averaged qualitative forms of the phase trajectory.

The following examples also will illustrate the applicability of the averaged forms of the evolution functions to predict the qualitative behavior of phase trajectories in more complicated conditions at the action of harmonic torques, and also at the increasing of magnitudes of torques.

Example 3.

$$M_x = M_y = 0, \;\; M_z = 20C(0)\sin(2t).$$

The inertia moments are described by (60).

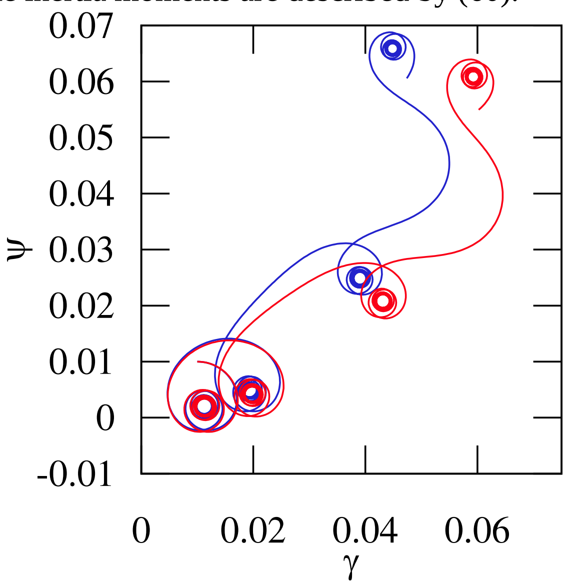

Fig. 9 The thrust vector hodograph evolution of Example 3

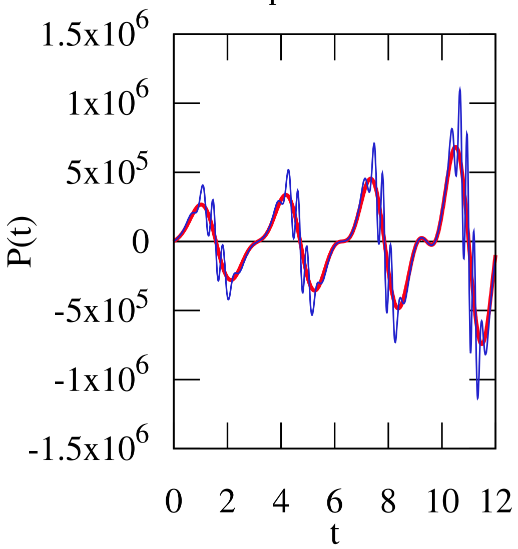

Fig. 10 The curvature function *P(t)* of Example 3

Example 4.

$$M_x = M_y = 0, \;\; M_z = 30C(0)\sin(2t).$$

The inertia moments are described by (60).

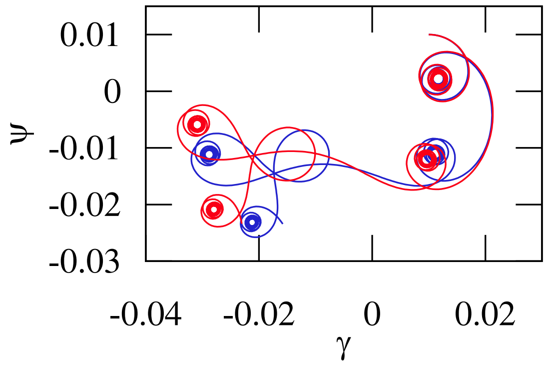

Fig. 11 The thrust vector hodograph evolution of Example 4

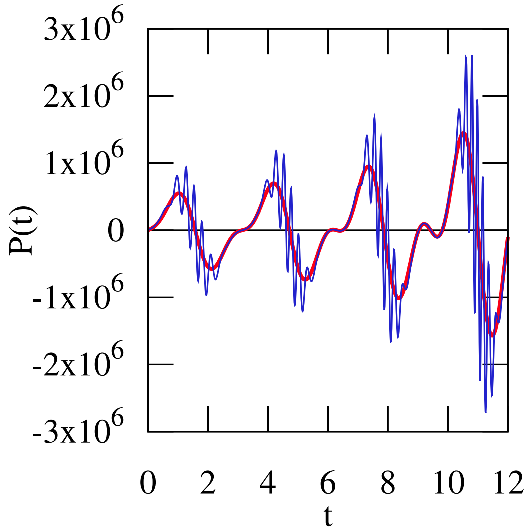

Fig. 12 The curvature function *P(t)* of Example 4

Example 5.

$$M_x = 0.1A(0), \;\; M_y = 0, \; M_z = 30C(0)\sin(2t).$$

The inertia moments are described by (60).

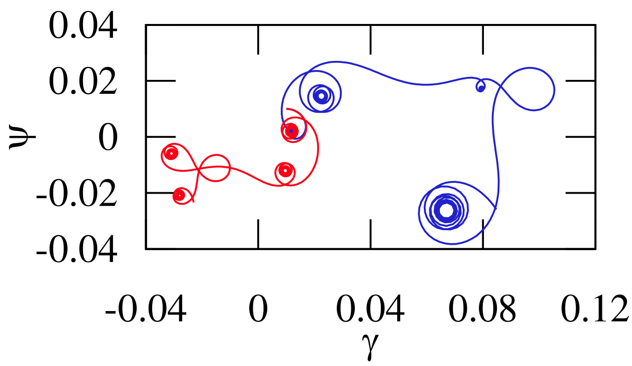

Fig. 13 The thrust vector hodograph evolution of Example 5

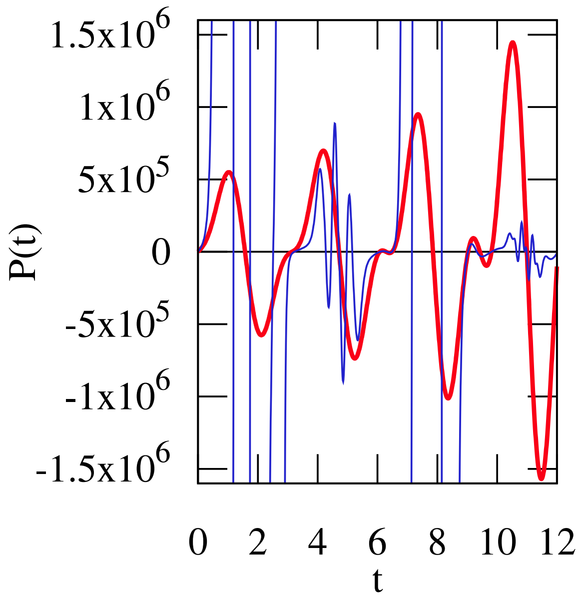

Fig. 14 The curvature function *P(t)* of Example 5

As multiple examples show, the adequate correspondence of the direct and the averaged trajectories is presented in aspects of qualitative properties of the motion. Indeed there are some relative deviations in forms, but they do not break the main idea and the suggested method of qualitative analysis/synthesis of the dynamical behavior of the spacecraft with variable inertia-mass parameters in aspects of its nutational-precessional motion.

Examples 3 and 4 well demonstrate all the features of the method application at the action of oscillating external torques. We still obtain the adequate correspondence of direct and averaged results at the complication of the motion conditions.

The final example 5 represents the most difficult type of analysis at the simultaneous action of both perturbing factors (oscillating torques and constant torques). On the one hand, we see a significant divergence of the curves (direct and averaged), however, on the other hand, these curves are completely equivalent in the topological sense: we see the same number and alternation order of twisted and untwisted zones.

Thus, the presented examples confirm the possibility of using the generalized method of the analysis of averaged curvature of phase trajectories to predict and to synthesize the qualitative properties of regimes of variable mass spacecraft attitude dynamics.

7. Conclusion

In the paper, the attitude dynamics of the variable mass spacecraft was investigated with the help of the qualitative method for non-autonomous dynamical systems analysis. This method was developed earlier [26, 27] for the case of motion of dynamically symmetrical rigid bodies with variable mass. In this paper, this method was generalized for the dynamically asymmetric spacecraft case. The generalized form of the method allowed to obtain the conditions for predefined motion modes of the variable mass spacecraft with an asymmetrical inertia-mass configuration.

Some special examples are considered that illustrate the efficiency of the method and the topological equivalence between the original system and the averaged one.

The phase trajectory curvature evaluation gives an opportunity to synthesize the appropriate time-dependencies of inertia-mass parameters of spacecraft, which allow to decrease the nutation by passive way and, therefore, to increase the accuracy of the spin-stabilization and the thrust impulse formation during operation of the jet-rocket engine.

Acknowledgments

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