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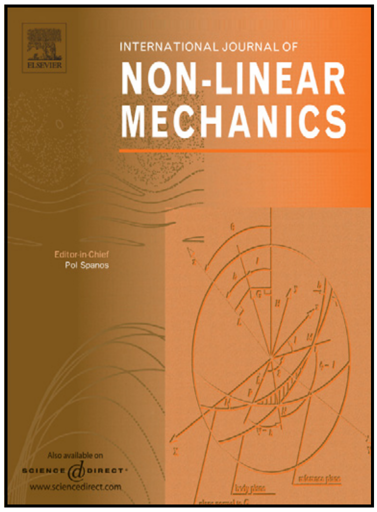

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PII: S0020-7462(12)00171-0
DOI: http://dx.doi.org/10.1016/j.ijnonlinmec.2012.10.012
Reference: NLM2078

To appear in: *International Journal of Non-Linear Mechanics*

Received date: 17 April 2012
Revised date: 28 September 2012
Accepted date: 27 October 2012

Cite this article as: Anton V. Doroshin, Exact solutions for angular motion of coaxial bodies and attitude dynamics of gyrostat-satellites, *International Journal of Non-Linear Mechanics,* http://dx.doi.org/10.1016/j.ijnonlinmec.2012.10.012

EXACT SOLUTIONS FOR ANGULAR MOTION OF COAXIAL BODIES AND ATTITUDE DYNAMICS OF GYROSTAT-SATELLITES

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Abstract. Dynamics of the torque-free angular motion of gyrostat-satellites and dual-spin spacecraft are examined. New analytical solutions for the angular moment components are obtained in terms of Jacobi elliptic functions. Also analytical solutions for Euler's angles are found. These solutions can be used for a dual-spin spacecraft and gyrostat-satellites attitude dynamics analysis and synthesis.

Keywords: Dual-Spin Spacecraft; Angular Motion; Polhode; Explicit Solutions; Elliptic Integrals

MSC2010: 70E17, 34C28, 37D45

Introduction

Study of the angular motion of rigid bodies and attitude dynamics of gyrostat-satellites (GS) and dual-spin spacecraft (DSSC) still remains one of the main research areas of modern mechanics and spaceflight dynamics. This research area connected with classical tasks of angular rigid body motion, gyrostats and coaxial bodies' systems motion [1-6]. Regular and irregular (chaotic) motion modes, attitude control and reorientation of gyrostats and coaxial spacecraft are being examined by many scientists [7-40].

Classical investigation results of rigid body and gyrostats motion have been presented in many works, for example, in [1-7]. In [8, 9] important aspects of torque-free motion dynamics of gyrostats were studied.

Analysis of angular motion of coaxial bodies and dual-spin spacecraft was conducted, for example, in [10-20] including perturbed cases of motion. In works [17-19] research results for DSSC motion at rotor-body spinup realization (a momentum transfer maneuver) were collected. In [20-33] compound and chaotic modes of motion of gyrostats and DSSC at absence/presence of control were considered.

In work [34] angular motion of variable mass dual-spin spacecraft was investigated. Corresponded research results showed non-trivial changes of DSSC angular motion at the variability of mass–inertia parameters. Also the qualitative method for phase space analysis based on the evaluation of a phase trajectory curvature was developed – this method can be used for the synthesis of realization conditions of special motion modes (for example, monotone decrease/increase of nutation angle).

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In paper [35] a chaotic motion of gyrostats in resistant environment was considered with the help of well-known dynamical systems with strange attractors: Lorenz, Rössler, Newton–Leipnik and Sprott systems. Cases for perturbed gyrostats motion with variable periodical inertia moments and with periodical internal rotor relative angular moment was investigated.

In [36] Heteroclinic dynamics of the torque-free dual-spin spacecraft was examined, and analytical solutions for heteroclinic orbits, corresponded to the polhodes-separatrices in the space of the angular moment components, was obtained. Analysis of possibility of the motion chaotization with the help of Melnikov method was conducted on the base of these analytical heteroclinic solutions.

In works [37-40] interesting recent results for gyrostat’s type systems can be found. These results connected with investigation of new aspects of stabilization and synchronization of electro-mechanical gyrostat systems (in regular and chaotic cases), dynamics of gyrostats in the gravitational field, and gyrostats' reorientation problems.

In the framework of gyrostats dynamics we need to emphasize the analytical study problem. Analytical exact solutions for gyrostats motion have an important value as intrinsic mathematical and mechanical problem, and also these results may be considered as unperturbed generating dependences at investigation of the perturbed motion's dynamics at presence of small external and internal torques, such as gravity gradient influence, geomagnetic field torques, aerodynamic moments, interactions of DSSC bodies, etc.

The main analytical investigations and exact solutions for the parameters of balanced gyrostat motion are presented in works [1-4, 7]; also explicit solution for gyrostat was recently found in [8]. Obtaining of analytical results for non-Kelvin-type gyrostats and dual-spin spacecraft motion was considered in [9-12]. Here particularly we need to underscore the exact explicit solutions for DSSC angular motion presented in [10] and replicated in [11, 12] – these important solutions were obtained for the Andoyer–Deprit canonical variables in closed form in Jacobi elliptic functions.

So, in this paper on the base of Euler’s dynamical equations we obtain a new form of the exact explicit solutions for the coaxial system and DSSC in the space of angular velocity components.

The paper is organized as follows. In section 1 the mathematical model of the coaxial bodies system, GS and DSSC is presented. In section 2 the explicit analytical solutions for the angular velocity (angular moment) components are obtained. In section 3 analytical solutions for Euler’s angles are found.

1. The motion equations of the coaxial bodies, GS and DSSC

Let us consider the torques free attitude dynamics of the coaxial bodies, GS and DSSC which was started in [36]. GS and DSSC consist of two coaxial bodies (body #1 is a rotor; body #2 is a main/core/carrier body). We assume that the main body has triaxial inertia tensor and the rotor is dynamically symmetrical body. Let us introduce following coordinate frames (Fig.1): $OXYZ$ is the inertial system of coordinates, $Ox_2y_2z_2$ – the connected principal system of coordinates of the carrier body, and $Ox_1y_1z_1$ – the connected principal system of coordinates of the rotor body. The axes Oz_1 and Oz_2 of the connected systems are identical to the common rotation axis of the coaxial bodies.

The system motion can be described on the base of Euler dynamical equations [3, 4], and with the help of Andoyer–Deprit canonical variables [13, 14]. The dynamical Euler's equation of the torque-free motion of the coaxial system can be written, for example [34, 36], as:

$$\begin{cases} A\dot{p} + (C - B)qr + qC_1\sigma = 0 \\ B\dot{q} + (A - C)pr - pC_1\sigma = 0 \\ C\dot{r} + C_1\dot{\sigma} + (B - A)pq = 0 \\ C_1(\dot{r} + \dot{\sigma}) = M_\Delta = 0 \end{cases} \tag{1.1}$$

or in the following form:

$$\begin{gathered} A\dot{p} + (C_2 - B)qr + q\Delta = 0; \quad B\dot{q} + (A - C_2)pr - p\Delta = 0 \\ C_2\dot{r} + \dot{\Delta} + (B - A)pq = 0; \quad \dot{\Delta} = M_\Delta = 0 \end{gathered} \tag{1.2}$$

where $\{p,q,r\}$ are components of the carrier body's angular velocity which represented in projections onto the axes of the $Ox_2y_2z_2$ frame; $\sigma -$ the rotor angular velocity relatively the carrier body; $\mathbf{I}_2 = diag[A_2, B_2, C_2]$ is the triaxial inertia tensor of the carrier body in the connected frame $Ox_2y_2z_2$; $\mathbf{I}_1 = diag[A_1, A_1, C_1]$ is the inertia tensor of the dynamically symmetrical rotor in the connected frame $Ox_1y_1z_1$; $A = A_1 + A_2$, $B = A_1 + B_2$, $C = C_1 + C_2$ are the main inertia moments of the coaxial bodies system in the frame $Ox_2y_2z_2$ (including rotor); $M_\Delta -$ is the internal torque of the coaxial bodies interaction (assume $M_\Delta = 0$); $\Delta = C_1(r + \sigma) -$ the longitudinal angular moment of the rotor along Oz_1; $C_1\sigma = h_{z_1} -$ the rotor relative angular moment in the carrier body frame $Ox_2y_2z_2$. We assume following conditions $A_2 > B_2 > C_2 > A_1 > C_1, \quad \Delta = \text{const} > 0$.

Also we need to add the well-known kinematical equations for Euler's angles:

$$\begin{cases} \dot{\psi} = \dfrac{1}{\sin\theta}(p\sin\varphi + q\cos\varphi) \\ \dot{\theta} = p\cos\varphi - q\sin\varphi \\ \dot{\varphi} = r - \dfrac{\cos\theta}{\sin\theta}(p\sin\varphi + q\cos\varphi) \\ \dot{\delta} = \sigma \end{cases} \tag{1.3}$$

Here we note that the equation system (1.1) corresponds to the torque-free motion of the coaxial bodies and the unbalanced gyrostat with non-constant relative angular moment of rotor ($h_{z_1} = C_1\sigma \neq \text{const}$, even if $M_\Delta = 0$). In this case results of analysis for Kelvin-type gyrostats [1-4, 7-8] are not applicable.

Also we can use the Hamiltonian form of equations in the Andoyer–Deprit canonical variables. The Andoyer–Deprit variables [5, 9, 10, 33, 36] can be expressed with the help of the coaxial system's angular moment $\mathbf{K}$ (fig.1):

$$L = \frac{\partial T}{\partial \dot{l}} = \mathbf{K}\cdot\mathbf{k}; \ I_2 = \frac{\partial T}{\partial \dot{\varphi}_2} = \mathbf{K}\cdot\mathbf{s} = |\mathbf{K}| = K; \ I_3 = \frac{\partial T}{\partial \dot{\varphi}_3} = \mathbf{K}\cdot\mathbf{k}'; \quad L \le I_2$$

$$K_{x_2} = Ap = \sqrt{I_2^2 - L^2}\sin l; \; K_{y_2} = Bq = \sqrt{I_2^2 - L^2}\cos l; \; K_{z_2} = C_2 r + \Delta = L. \tag{1.4}$$

The system Hamiltonian [9, 33, 36] in the Andoyer–Deprit phase space takes the form:

$$H = \frac{I_2^2 - L^2}{2}\left[\frac{\sin^2 l}{A_1 + A_2} + \frac{\cos^2 l}{A_1 + B_2}\right] + \frac{1}{2}\left[\frac{\Delta^2}{C_1} + \frac{(L-\Delta)^2}{C_2}\right] + \varepsilon H_1, \tag{1.5}$$

where T – is the system kinetic energy; εH_1 – is the small perturbed part of the Hamiltonian, connected with such disturbances as gravity gradient, geomagnetic field etc. and corresponded small potentials. In this research we focus only on the unperturbed $(\varepsilon = 0)$ generating solutions obtaining and, therefore, effects of small perturbations can be eliminated.

Using the Hamiltonian (1.5), we can write corresponded dynamical system

$$\begin{cases} \begin{cases} \dot{L} = -\dfrac{\partial H}{\partial l} = \alpha\left(I_2^2 - L^2\right)\sin l \cos l; \\ \dot{l} = \dfrac{\partial H}{\partial L} = L\left[\dfrac{1}{C_2} - \dfrac{\sin^2 l}{(A_1 + A_2)} - \dfrac{\cos^2 l}{(A_1 + B_2)}\right] - \dfrac{\Delta}{C_2}; \end{cases} & \begin{cases} \dot{I}_2 = -\dfrac{\partial H}{\partial \varphi_2} = 0 \Rightarrow I_2 = \text{const}; \\ \dot{\varphi}_2 = \dfrac{\partial H}{\partial I_2} = I_2\left[\dfrac{\sin^2 l}{A_1 + A_2} + \dfrac{\cos^2 l}{A_1 + B_2}\right]; \end{cases} \\ \begin{cases} \dot{I}_3 = -\dfrac{\partial H}{\partial \varphi_3} = 0 \Rightarrow I_2 = \text{const}; \\ \dot{\varphi}_3 = \dfrac{\partial H}{\partial I_3} = 0 \Rightarrow \varphi_3 = \text{const}; \end{cases} & \begin{cases} \dot{\Delta} = -\dfrac{\partial H}{\partial \delta} = 0 \Rightarrow \Delta = \text{const}; \\ \dot{\delta} = \sigma = \dfrac{\partial H}{\partial \Delta} = \dfrac{\Delta}{C_1} - \dfrac{(L-\Delta)}{C_2}. \end{cases} \end{cases} \tag{1.6}$$

where $\alpha = (A_1 + B_2)^{-1} - (A_1 + A_2)^{-1}$.

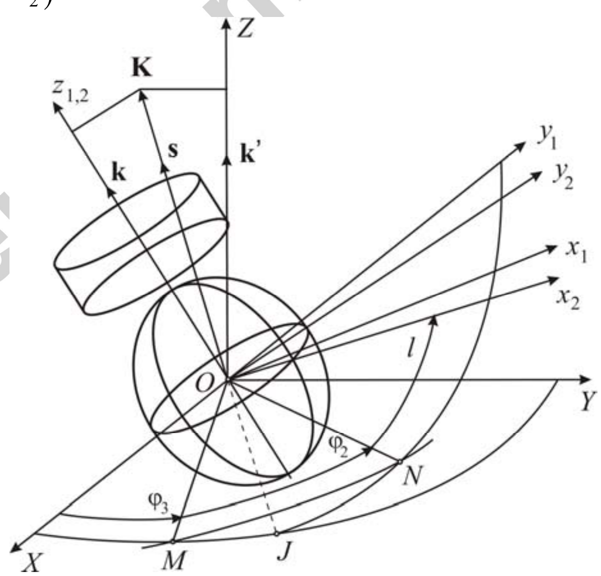

Fig.1. *The DSSC (coaxial bodies system) and the coordinate frames*

As we can see, the system (1.6) consists of four subsystems for every pair of canonical Andoyer–Deprit variables $\{\{l,L\},\{\varphi_2,I_2\},\{\varphi_3,I_3\},\{\delta,\Delta\}\}$. The subsystem $\{l,L\}$ is independent and can be integrated separately. This integration in elliptical quadratures was performed in paper [10].

We need to note that truncation of the subsystems $\{\varphi_2,I_2\},\{\varphi_3,I_3\},\{\delta,\Delta\}$ inevitably leads to information loss. For example, comparison of the third-order differential equations system (1.2) ($\Delta = \mathrm{const}$) with the second-order subsystem $\{l,L\}$ (1.6) is identical to the comparison of 3D geometrical object and its plane projection – for our mechanical task dynamical system orbit represents 3D-polhode, and it should be described with the help of three parameters in 3D-space {p, q, r} instead of the consideration based on its projection on the Andoyer–Deprit phase plane. Therefore, reduction $\{p,q,r\} \to \{l,L\}$ is easily executable, but inverse transformation $\{l,L\} \to \{p,q,r\}$ without information loss is impossible (it will be shown at the end of the section 2). Moreover, to give complete description of the motion we need to take into account such kinematical parameters as Euler's angles. Thus, the full dynamical system corresponds to the mechanical system for coaxial bodies (GS, DSSC) with four degree of freedom (4-DOF).

In the next section we will obtain analytical solutions for angular velocity components of the coaxial bodies system.

2. Explicit analytical solutions for angular moment components

The term “polhode” is well-known [3]. The polhode is the fourth-order curve in 3D-space (Fig.2) which corresponds to the intersection of a kinetic energy ellipsoid and an angular moment ellipsoid, which are defined with the help of the following expressions [3]:

$$Ap^2 + Bq^2 + C_2 r^2 + \frac{\Delta^2}{C_1} = 2T \tag{2.1}$$

$$A^2 p^2 + B^2 q^2 + \left[C_2 r + \Delta\right]^2 = K^2 = 2DT \tag{2.2}$$

$$D = \frac{K^2}{2T} \tag{2.3}$$

We can write polhode equation [3, 36] on the base of expressions (2.1) and (2.2) combinations. So, multiplication (2.1) by A and deduction (2.2) gives

$$B\left(A-B\right)q^2 + C_2\left(A-C_2\right)r^2 + C_1\left(A-C_1\right)\left(r+\sigma\right)^2 - 2C_1C_2 r\left(r+\sigma\right) = 2T\left(A-D\right) \tag{2.4}$$

Multiplication (2.1) by C_1 and deduction (2.2) gives

$$A\left(C_1-A\right)p^2 + B\left(C_1-B\right)q^2 + C_2\left(C_1-C_2\right)r^2 - 2C_1C_2 r\left(r+\sigma\right) = 2T\left(C_1-D\right) \tag{2.5}$$

Multiplication (2.1) by B and deduction (2.2) gives

$$A\left(B-A\right)p^2 + B\left(C_2 r^2 + C_1\left[r+\sigma\right]^2\right) - \left(C_2 r + C_1\left[r+\sigma\right]\right)^2 = 2T\left(B-D\right) \tag{2.6}$$

Taking into account perfect square, equation of hyperbolae (on the coordinate plane *Opr* at the Fig.2) follows from (2.6)

$$-A(A-B)p^2 + C_2(B-C_2)\left[r-\frac{\Delta}{B-C_2}\right]^2 = F \tag{2.7}$$

where $F = 2T(B-D)+\Delta^2 a, \qquad a = \dfrac{C_1C_2+(B-C_2)(C_1-B)}{(B-C_2)C_1}$

From expression (2.4) equation of ellipses (on the coordinate plane *Oqr* (fig.2)) follows

$$B(A-B)q^2 + C_2(A-C_2)\left[r-\frac{\Delta}{A-C_2}\right]^2 = H \tag{2.8}$$

$$b = \frac{C_2C_1+(A-C_2)(C_1-A)}{(A-C_2)C_1} \qquad H = 2T(A-D)+\Delta^2 b$$

Equation (2.7) can be rewritten as

$$p = \pm\sqrt{\frac{C_2(B-C_2)\left[r-\frac{\Delta}{B-C_2}\right]^2 - F}{A(A-B)}} \tag{2.9}$$

From (2.8) we can get

$$r-\frac{\Delta}{A-C_2} = \pm\sqrt{\frac{H-B(A-B)q^2}{C_2(A-C_2)}} \tag{2.10}$$

With the help of (2.10) we can write

$$r-\frac{\Delta}{B-C_2} = \pm\sqrt{\frac{H-B(A-B)q^2}{C_2(A-C_2)}} - \Delta\beta \tag{2.11}$$

where $\beta = (B-C_2)^{-1} - (A-C_2)^{-1}$.

On the base of (2.11) and (2.9) we can rewrite the second equation (1.2) in the form

$$B\dot{q} = \mp\sqrt{\frac{C_2(B-C_2)}{A(A-B)}\left[\pm\sqrt{\frac{H-B(A-B)q^2}{C_2(A-C_2)}} - \Delta\beta\right]^2 - \frac{F}{A(A-B)}} \cdot \sqrt{\frac{A-C_2}{C_2}\left[H-B(A-B)q^2\right]} \tag{2.12}$$

We can make the change of variables (case 1)

$$x = +\sqrt{\frac{H-B(A-B)q^2}{C_2(A-C_2)}} - \Delta\beta \tag{2.13}$$

Then from (2.13) we obtain

$$(x+\Delta\beta)^2 = \frac{1}{C_2(A-C_2)}\left[H-B(A-B)q^2\right] \tag{2.14}$$

$$q = \pm\sqrt{\frac{H-C_2(A-C_2)(x+\Delta\beta)^2}{B(A-B)}}; \qquad dq = \mp\frac{C_2(A-C_2)}{B(A-B)}\frac{(x+\Delta\beta)\,dx}{\sqrt{\frac{H-C_2(A-C_2)(x+\Delta\beta)^2}{B(A-B)}}} \tag{2.15}$$

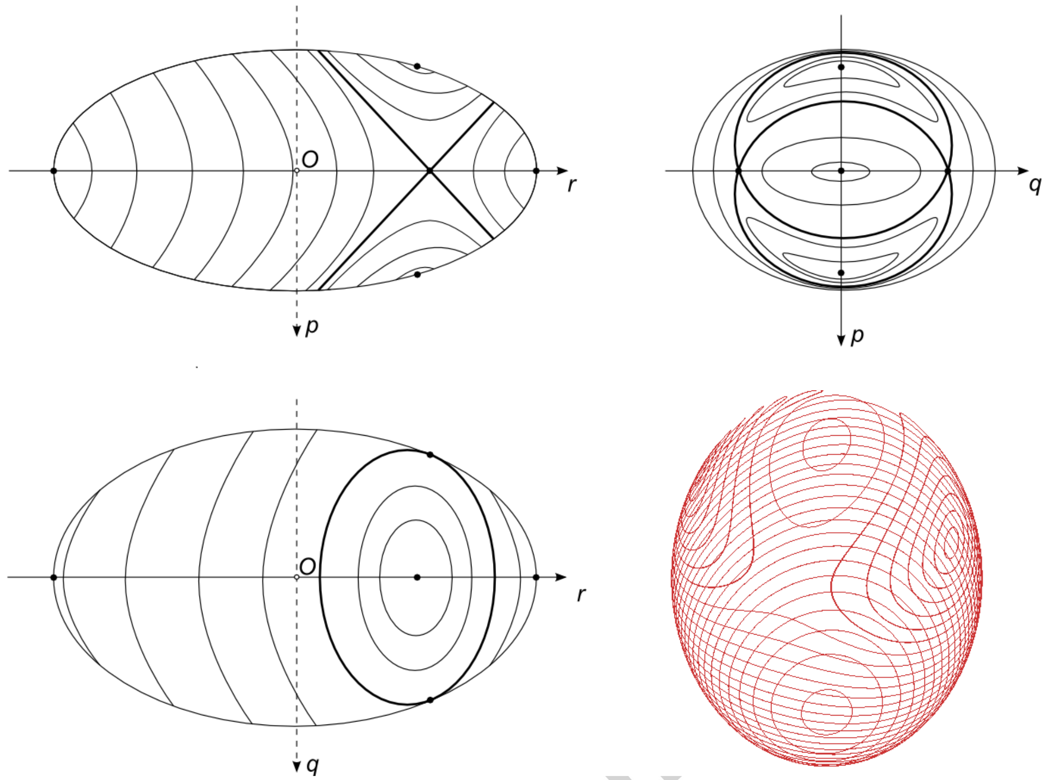

Fig.2 *Projections and 3D-image of the polhodes ellipsoid*

Also we can make the following change of variables (case 2)

$$x = -\sqrt{\frac{H - B(A-B)q^2}{C_2(A-C_2)}} - \Delta\beta \tag{2.16}$$

Similar to previous case (case of change (2.13)) from (2.16) expressions (2.14) and (2.15) follow again. Thus, for both changes we have interconnected equation

$$dt = \pm\frac{M\,dx}{\sqrt{H - a(x+b)^2}\sqrt{cx^2 - G}} \tag{2.17}$$

where

$$M = C_2\sqrt{\frac{B}{A-B}};\quad G = \frac{F}{A(A-B)};\quad a = C_2(A-C_2);\quad b = \Delta\beta;\quad c = \frac{C_2(B-C_2)}{A(A-B)}$$

It is needed to note differences between initial values corresponded to cases (i=1, 2):

$$x(t_0) = x_i^{ini} = (-1)^{i+1}\sqrt{\frac{H - B(A-B)q_0^2}{C_2(A-C_2)}} - \Delta\beta,\quad i = 1,2$$

Let us rewrite equation (2.17) in the form with differences of squares

$$dt = \pm\frac{M}{\sqrt{ac}}\frac{dx}{\sqrt{\left(\sqrt{\frac{H}{a}}\right)^2 - (x+b)^2}\sqrt{x^2 - \left(\sqrt{\frac{G}{c}}\right)^2}} \tag{2.18}$$

Now we make the next change of variables

$$z = \sqrt{\frac{R(x-e)}{P(x+e)}}$$
$$R = -b - d + e; \quad P = -b - d - e \tag{2.19}$$
$$d = \sqrt{H/a}; \quad e = \sqrt{G/c}$$

Then we have the following expressions

$$x = \frac{-R/P - z^2}{-R/P + z^2} e; \qquad dx = \frac{R}{P} \frac{4ez\,dz}{\left(-R/P + z^2\right)^2} \tag{2.20}$$

Substitution (2.20) into (2.18) gives us

$$dt = \pm \frac{4eMR}{P\sqrt{ac}} \left[\sqrt{\frac{H}{a}\left(z^2 - \frac{R}{P}\right)^2 - \left(e\left[\frac{-R}{P} - z^2\right] + b\left[z^2 - \frac{R}{P}\right]\right)^2} \sqrt{e^2\left(\frac{-R}{P} - z^2\right)^2 - \frac{G}{c}\left(z^2 - \frac{R}{P}\right)^2} \right]^{-1} zdz$$

After some transformations with expansion of difference of squares last equation takes the form

$$dt = \pm 2eM \frac{\sqrt{R/P}}{\sqrt{aG}} \left[\sqrt{\left(s_1 z^2 - s_2\right)\left(s_3 z^2 - s_4\right)} \right]^{-1} dz$$
$$s_1 = \sqrt{\frac{H}{a}} + e - b; \; s_2 = \frac{R}{P}\left[\sqrt{\frac{H}{a}} - e - b\right] \tag{2.21}$$
$$s_3 = \sqrt{\frac{H}{a}} - e + b; \; s_4 = \frac{R}{P}\left[\sqrt{\frac{H}{a}} + e + b\right]$$

Now (2.21) can be rewritten

$$dt = \pm 2eM \frac{\sqrt{R/P}}{\sqrt{aG}} \left[\sqrt{s_2 s_4} \sqrt{\left(1 - \frac{z^2}{c_1^2}\right)\left(1 - \frac{z^2}{c_2^2}\right)} \right]^{-1} dz \tag{2.22}$$

where $c_1^2 = s_2/s_1$; $c_2^2 = s_4/s_3$.

Here we should consider two cases of reduction of (2.22) to elliptic integral of the first kind:

1). If $c_1 < c_2$ then the following substitution is efficient: $z = c_1 y$ and, moreover, $k = c_1/c_2 < 1$.

2). If $c_1 > c_2$ then: $z = c_2 y$ and $k = c_2/c_1 < 1$.

The equation (2.22) takes the form

$$dt = \pm 2eM \frac{\sqrt{R/P}}{\sqrt{aG}} \frac{c_j dy}{\sqrt{s_2 s_4}\sqrt{\left(1-y^2\right)\left(1-k^2 y^2\right)}}; \quad j = 1, 2 \tag{2.23}$$

where index *j* corresponds to number of reduction case.

Integration gives us

$$\pm\left[N\left(t - t_0\right) + I_0\right] = \int_0^y \frac{dy}{\sqrt{\left(1-y^2\right)\left(1-k^2 y^2\right)}}$$
$$N = \left[2eM \frac{c_j \sqrt{R/P}}{\sqrt{aG}\sqrt{s_2 s_4}}\right]^{-1}; \quad I_0 = \int_0^{y_0} \frac{dy}{\sqrt{\left(1-y^2\right)\left(1-k^2 y^2\right)}} = \text{const} \tag{2.24}$$

Inversion of elliptic integral gives the explicit analytical solution

$$y(t) = \operatorname{sn}\left[\pm\left(N(t-t_0)+I_0\right),k\right] \tag{2.25}$$

where sn(u,k) is the Jacobi elliptic sine function with classical definition:

$$u = \Phi(\varphi) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1-k^2\sin^2(\vartheta)}}; \quad \varphi = \varphi(u) = \Phi^{-1}(u); \quad \operatorname{sn}(u,k) = \sin(\varphi(u));$$

After inverse transformations we obtain the exact explicit analytical solutions for all angular velocity components

$$\begin{cases} q(t) = \pm\sqrt{\dfrac{1}{B(A-B)}\left[H-\{x(t)+\Delta\beta\}^2 C_2(A-C_2)\right]} \\ p(t) = \pm\sqrt{\dfrac{1}{A(A-B)}\left[C_2(B-C_2)x^2(t)-F\right]} \\ r(t) = \dfrac{\Delta}{A-C_2} \pm (x(t)+\Delta\beta) \\ \sigma(t) = \dfrac{\Delta}{C_1} - r(t) = \dfrac{A-C}{C_1(A-C_2)}\Delta \mp (x(t)+\Delta\beta) \end{cases} \tag{2.26}$$

where

$$x(t) = e\frac{R/P+\operatorname{sn}^2\left[\pm\left(N(t-t_0)+I_0\right),k\right]}{R/P-\operatorname{sn}^2\left[\pm\left(N(t-t_0)+I_0\right),k\right]} \tag{2.27}$$

Fig. 3 demonstrates the validity of solutions (2.26) - we see comprehensive coincidence of the analytical (points) and numerical integration results (lines).

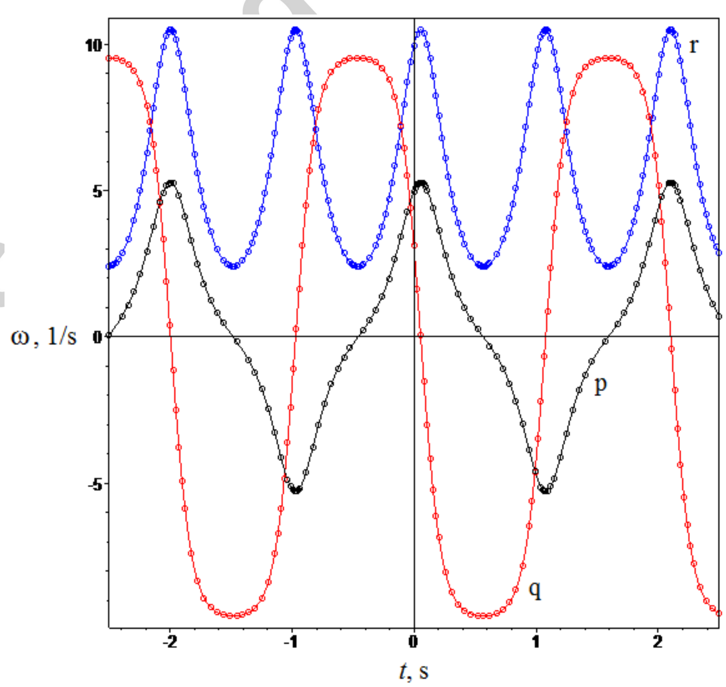

Fig.3 *Numerical (lines) and analytical (points) integration results*
A_2=15; B_2=8; C_2=6; A_1=5; C_1=4; p_0=5; q_0=5; r_0=10; $\Delta = 5$

It is well-known fact that solutions for gyrostats (coaxial bodies) generalize results for the free rigid body in Euler's case of motion [1-8]. In contrast to [1-8], in this paper the essential motion of the coaxial system (DSSC) was investigated without assumption of constancy of the gyrostatic moment ($h_z = C_1\sigma \neq \text{const}$).

In [10] analytical solutions for DSSC were obtained in terms of Jacobi elliptic functions for the Andoyer–Deprit canonical variables. With the help of [10] and based on kinematic expressions (1.4) we can also write the dependences $\{p(t), q(t), r(t)\}$ as functions of Deprit's variable *L*. In this case expressions shape includes compositions of Jacobi elliptic functions, radicals, trigonometric functions and inverse trigonometric functions at the same time – this structural complexity connected with information loss at the transformation of 3D-polhodes to their 2D-images on the $\{l, L\}-$plane:

$$\begin{aligned}
p(t) &= \frac{I_2}{A}\sqrt{1-s^2(t)}\sin\left[\pm\frac{1}{2}\arccos\left(\frac{(a'+b'-2)s^2(t)+4d's(t)+4h'-a'-b'}{\left(1-s^2(t)\right)(b'-a')}\right)\right] \\
q(t) &= \frac{I_2}{B}\sqrt{1-s^2(t)}\cos\left[\pm\frac{1}{2}\arccos\left(\frac{(a'+b'-2)s^2(t)+4d's(t)+4h'-a'-b'}{\left(1-s^2(t)\right)(b'-a')}\right)\right] \\
r(t) &= \left(L(t)-\Delta\right)/C_2; \qquad L(t) = s(t)I_2; \\
s(\tau) &= \frac{s'_2 s'_{31} - s'_3 s'_{21} sn^2(\omega\tau, k)}{s'_{31} - s'_{21} sn^2(\omega\tau, k)}; \qquad \tau = t \cdot I_2 / C_1; \qquad a', b', d', h', s'_i, s'_{ij} - \text{const}
\end{aligned} \tag{2.28}$$

We additionally note that results (2.28) follow directly from work [10].

Reduction of the solutions (2.28) to the new form (2.26) by analytical transformations is problematically. Vice versa, based on (2.26) we can write simple solutions for Andoyer–Deprit variables:

$$L(t) = C_2 r(t) + \Delta; \quad l(t) = \arcsin\frac{Ap(t)}{\sqrt{K^2 - \left(C_2 r(t) + \Delta\right)^2}} = \arccos\frac{Bq(t)}{\sqrt{K^2 - \left(C_2 r(t) + \Delta\right)^2}} \tag{2.29}$$

3. Explicit analytical solutions for Euler's angles

Let us consider angular motion of the coaxial system (DSSC) with respect to the initial frame *OXYZ* (Fig.1) in the case when vector of angular moment **K** is directed along *OZ*. It always can be realized by the changing of coordinate frames. In this case nutation angle (angle between *OZ* and $Oz_{1,2}$ axes) and intrinsic rotation angle (φ) can be found using classical method [3, 5] from expressions

$$\begin{cases}
K_{x_2} = Ap = K\sin\theta\sin\varphi \\
K_{y_2} = Bq = K\sin\theta\cos\varphi \\
K_{z_2} = C_2 r + \Delta = K\cos\theta
\end{cases} \tag{3.1}$$

So, taking into account analytical solutions for angular velocity components (2.26) we obtain exact solutions for angles

$$\cos\theta(t) = \left[C_2 r(t) + \Delta\right]/K; \quad \mathrm{tg}\,\varphi(t) = \frac{Ap(t)}{Bq(t)} \tag{3.2}$$

In considered case $(K = I_2 = I_3)$ with the help of comparison of expressions (3.1) and (1.4) we can write the correspondences between Euler and Andoyer–Deprit variables

$$\cos\theta = L/I_2; \; l = \varphi \tag{3.3}$$

Solutions for precession (ψ) and relative rotation angle (δ) follow from the first and the fourth equations (1.3) and expressions (3.1)

$$\psi(t) - \psi_0 = \int_{t_0}^{t} K \frac{Ap^2(t) + Bq^2(t)}{A^2 p^2(t) + B^2 q^2(t)} dt; \quad \delta(t) - \delta_0 = \int_{t_0}^{t} \left(\frac{\Delta}{C_1} - r(t)\right) dt \tag{3.4}$$

Fig. 4 demonstrates the validity of solution (3.2) for nutation angle (θ); Fig. 5 shows the validity of analytical solution for intrinsic rotation angle (φ) (3.2), for precession angle (ψ) (3.4) and for relative rotation angle (δ) (3.4). We note that all angles (Fig.4-5) were calculated in radians.

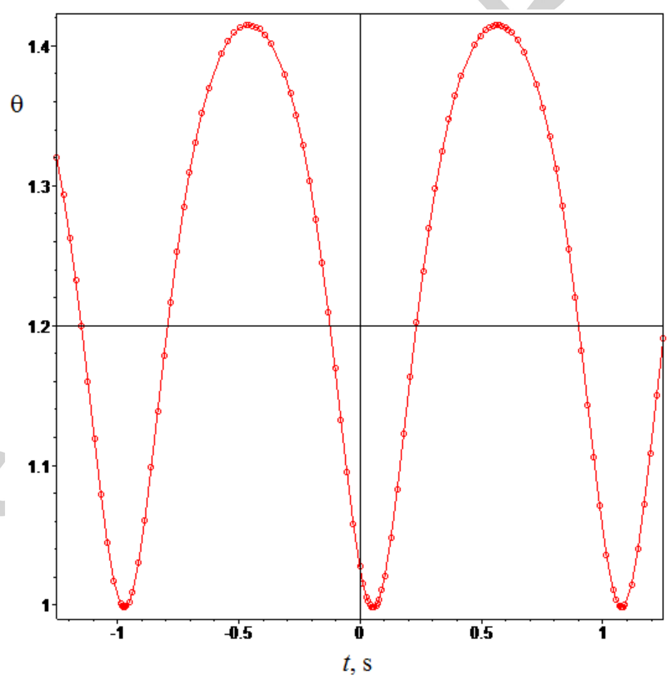

Fig.4 *Numerical (lines) and analytical (points) integration results for nutation*
A_2=15; B_2=8; C_2=6; A_1=5; C_1=4; p_0=5; q_0=5; r_0=10; $\Delta = 5$

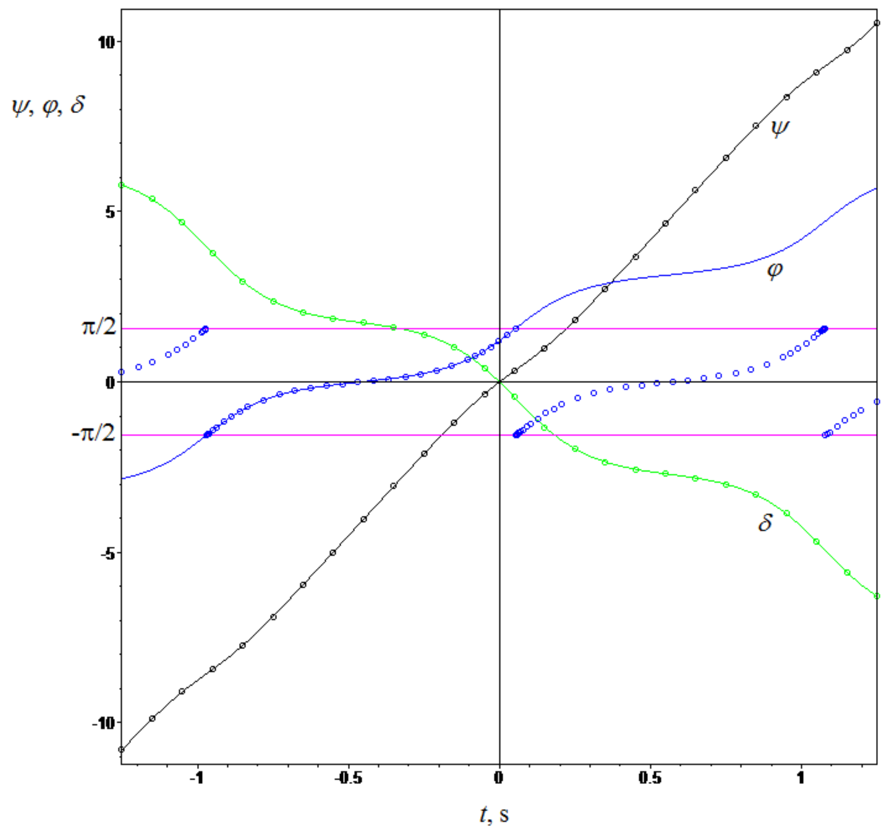

Fig.5 *Numerical (lines) and analytical (points) integration results for intrinsic rotation, precession and relative rotation angle*
A_2=15; B_2=8; C_2=6; A_1=5; C_1=4; p_0=5; q_0=5; r_0=10; $\Delta = 5$

We need to note that magnitude (analytical result) of intrinsic rotation angle (Fig.5) is located into the interval $\left[-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}\right]$ in compliance with actual range of arctangent-function:

$$\varphi = \text{arctg}\left(\frac{Ap(t)}{Bq(t)}\right)$$

By this reason we ought to add π to φ-value every rotational period.

Thus, all explicit exact analytical solutions are found for torque-free angular motion of 4-DOF $\{p,q,r,\sigma,\psi,\theta,\varphi,\delta\}$ coaxial bodies system.

These unperturbed generating solutions ((3.5), (3.2) and (3.4)) can be used for investigation of perturbed motion problems, such as the angular motion of DSSC with electromagnetic equipment in geomagnetic field, orbital motion of large DSSC taking into account influence of the gravity gradient, and also attitude dynamics DSSC in the perturbed environment and with chaotic behavior at presence of small external/internal disturbances.

Conclusion

Dynamics of the torque-free coaxial bodies system (GS, DSSC) has been examined in the space of angular velocity (angular moment) components. The new analytical solutions for all angular moment components have been obtained in terms of Jacobi elliptic functions. Also analytical solutions for Euler's angles have been found. These solutions can be used for dual-spin spacecraft and gyrostat-satellites attitude motion analysis and synthesis.

Acknowledgements

This research was supported by Russian Foundation for Basic Research (RFBR## 11-08-00794-a, 12-08-09202-mob_z) and the Russian Federation Presidential Program for the Support of Russian Scientists and Leading Scientific Schools (MK-1497.2010.8).

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Highlights

General dynamics of the free coaxial bodies system, gyrostat and dual-spin spacecraft is examined.

New analytical solutions for angular moment components are obtained in Jacobi elliptic functions.

Analytical solutions for Euler's angles are found.

The solutions can be used for dual-spin spacecraft and gyrostat-satellites motion analysis.