Analysis of attitude motion evolutions of variable mass gyrostats and coaxial rigid bodies system

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Abstract

This work involves the research of an angular motion of a mechanical system consisting of coaxial bodies of variable mass in a translating coordinate system. The author gives a theorem on the change in the angular momentum of a system of variable mass coaxial bodies with respect to translating axes. The research of motion dynamics is conducted using the example of two coaxial bodies. The modification of mass-inertia parameters of coaxial bodies causes nontrivial changes of system angular motion. The article describes a special developed method for qualitative phase space analysis, based on the evaluation of a phase trajectory curvature. The method suggested makes it possible to determine the phase trajectory shape and to synthesize conditions for special motion modes realization (for example, nutation angle monotonous diminution or magnification). The results obtained can be used to describe the motion of a coaxial dual-spin spacecraft, performing active maneuvers with a change in mass.

Keywords: Coaxial Bodies, Unbalanced Gyrostat, Dual-Spin Spacecraft, Variable Mass, Active Maneuvers, Attitude Motion, Qualitative Method, Phase Trajectory Curvature

1. Introduction

Research of attitude motion of a system of coaxial rigid bodies and gyrostats always was and still remains one of the important problems of theoretical and applied mechanics. The dynamics of the attitude motion of rotating rigid bodies and gyrostats is a classic mechanical research object. Basic aspects of such motion were studied by Euler, Lagrange, Kovalevskaya, Zhukovsky, Volterra, Wangerin, Wittenburg. The main results of the attitude motion research can be found in appropriate treatises [1-5].

However, the study of the dynamics of rotating bodies and gyrostats is still very important in modern science and engineering. Among the basic directions of modern research within the framework of the indicated problem it is possible to highlight the following points: deriving exact and approximated analytical and asymptotic solutions [1-5, 25, 28], research of a stability of motion conditions [6-16], the analysis of motion under the influence of external regular and stochastic disturbance, research of dynamic chaos [17-22], research of non-autonomous systems with variable parameters [23-30].

N. Ye. Zhukovsky studied the motion of a rigid body containing cavities filled with homogeneous capillary liquid. The research showed that the equations of motion in such case can be reduced to the equations of the attitude motion of a gyrostat. Also analytical solutions of some special modes of motion of a gyrostat were found.

The ordinary differential equations of a gyrostat attitude motion with constant angular momentum were solved analytically by Volterra. Volterra solution has generalized a similar analytical solution for a rigid body in case of Euler. In the works of Wangerin and Wittenburg solution of Volterra is reduced to the convenient parameterization expressed in elliptic integrals.

The analytical solution for attitude motion of heavy dynamically symmetric gyrostat, colligating a classic solution for a heavy solid body in the case of Lagrange, is given in a paper [25]. In the indicated published works solutions for all Euler angles (precession, nutation, and intrinsic rotation) are found in elliptic functions and integrals. Also modes of motion with constant and variable relative angular momentum of gyrostat rotor are considered.

The issues of the rotational motion dynamics of a gyrostat are very important for numerous applications such as the dynamics of satellite-gyrostats, spacecrafts and aircrafts.

The attitude dynamics of gyrostat satellites and dual-spin (double-rotation) satellites has been studied by a number of scientists [6-22]. Most of these efforts were aimed on finding the equilibrium states in the presence of external disturbance torques [6-9], on analysis of the stability of spinning satellites under energy dissipation [10-16]. Some authors recently have investigated bifurcation and chaos in the gyrostat satellites [17-22].

Despite above-mentioned wide spectrum of research results the stated problem still remains actual, especially for the variable structure rigid bodies systems and variable mass dual-spin spacecrafts.

The possibility of change in some mass and inertia parameters and the structure variability can be explained by the fact that a spacecraft (SC) performs active maneuvers with the use of the jet engine.

Any SC in orbit is affected by external disturbances of different kind, e.g. the solar radiation pressure, the gravity gradient torque, the magnetic torque caused by the Earth's magnetic field, or the aerodynamic torque due to the action of a resisting medium like the Earth's atmosphere. However all these external disturbances are not large in comparison with the jet engine thrust of the SC on the active motion stage (e.g. inter-orbital transfer, orbit correction, attitude reorientation). Moreover, variability of mass parameters (mass and moments of inertia) has a considerable influence on attitude dynamics. The change of the moments of inertia entails change of angular momentum, which is the basic characteristic of attitude motion. Thereupon mass (structure) variation is one of the primary factors determining attitude motion of a SC.

For the purposes of better understanding the essence of this problem it is important to give a brief overview of the main considered engineering peculiarities of SC's active motion. An SC in order to perform an active maneuver (e.g. inter-orbital transfer) should create a jet engine thrust and thus obtain acceleration or braking momentum $\Delta\mathbf{V}$ (reorbit/ deorbit burn).

This momentum should be generated exactly in a pre-calculated direction. Engine thrust is usually focused along the SC's longitudinal axis, therefore it is necessary to stabilize the longitudinal axis in order to ensure the accurate momentum generation. Stabilization of the longitudinal axis can be carried out in a gyroscopic mode when SC spins around the longitudinal axis which is oriented in the calculated direction.

Momentum generation is not instantaneous, it demands a continuous operation of the jet engine within several seconds (or minutes). During this period of time a SC performs two motions: trajectory motion of a center of mass and an angular motion around it. Such angular motion obviously changes the location of the longitudinal axis and, hence, a direction of thrust.

The time history of thrust direction strongly affects the value and direction of a transfer momentum deviation. Consequently, the transfer is performed to the orbit different from the desired one. There is a "scattering" of thrust (Fig. 1). Therefore, it is very important to take SC angular motion into account during the analysis of the powered trajectory motion.

It is necessary to obtain the angular motion which ensures that SC's longitudinal axis (and the thrust vector) performs precessional motion with monotonously decreasing nutation angle. Thus the longitudinal axis travels inside an initial cone of nutation and the thrust vector naturally comes nearer to an axis of a precession which is a desired direction of transitional momentum output ("is focused" along a necessary direction).

When the angular motion does not provide a monotonous decrease in nutation angle the longitudinal axis moves in a complicated way. In such case the thrust vector also performs complicated motion and "scatters" the transitional momentum. A transfer orbit scatters as well.

Among the works devoted to the rigid bodies systems with variable mass and inertia parameters it is possible to mark the following [17, 18, 23, 24, 28, 30]. The work [18] contains the analysis of chaotic behavior of a spacecraft with a double rotation and time-dependent moments of inertia during it's free motion. The main investigation results of variable mass system dynamics should be found in the monographies [23, 24]. These results include Ivan V. Meschersky theory of motion of bodies with variable mass, theory of "short-range interaction" and "solidification (freezing)".

The equations of variable mass dynamically symmetrical coaxial bodies system were developed in papers [28]. Also in [27] the attitude motion of coaxial bodies system and double rotation spacecraft with linear time-dependent moments of inertia were analyzed and conditions of motion with decreasing value of nutation were found. The results [27] can be used for the analysis of attitude motion of a dual-spin spacecraft with an active solid-propellant rocket engine.

Current paper represents continuation of the research described in [27-30] and is devoted to the dynamics of variable mass coaxial bodies systems, unbalanced gyrostats and dual-spin spacecrafts.

The paper has the following structure: Section 1 – introduction of the primary theoretical and physical background, Section 2 – mathematical definition of the coaxial bodies attitude motion problem in terms of the angular momentum, Section 3 – main equations of attitude motion of two variable mass coaxial bodies system and unbalanced gyrostat, Section 4 – development of research method for the attitude motion of variable mass coaxial bodies and unbalanced gyrostat, Section 5 – examples of analysis of the attitude motion of variable mass unbalanced gyrostat, Section 6 – conclusion.

2. Problem Definition

Below we will derive the equations of motion of a system of k coaxial rigid bodies of variable mass with respect to translating coordinate frame *OXYZ*. The motion of the system is analyzed with respect to the following coordinate systems (Fig. 2): $P\xi\eta\zeta$ is a system of coordinates, fixed in absolute space, *OXYZ* is a moving coordinate system with origin at the point *O*, the axes of which remain collinear with the axes of the fixed system during the whole time of motion, and $Ox_iy_iz_i$ are systems of coordinates with a common origin, rigidly connected to the *i*-th body (i = 1, 2, ... , k), rotating with respect to the system *OXYZ*. *OXYZ* system has its origin in a point lying on the common axis of rotation of the bodies and matching with the initial position of the centre of mass $\left(t = t_0 : \ C \equiv O\right)$. Points in the different parts of the system are distinguished by the body they belong to, and in all expressions they are indicated by the subscript ν_i (where i is a number of an appropriate body).

To construct the equations of motion we use the "short-range" hypothesis – particles which obtain a relative velocity when separated from the body no longer belong to the body and have no effect on it. In such case the theorem on the change in the angular momentum of a system of variable mass [23], written with respect to the fixed system of coordinates $P\xi\eta\zeta$, has the following form:

$$\frac{d\mathbf{K}_P}{dt} = \mathbf{M}_P^e + \mathbf{M}_P^R + \sum_{i=1}^{k} \mathbf{S}_i^e \,,$$
$$\mathbf{S}_i^e = \sum_{\nu_i} \mathbf{r}_{\nu_i} \times \frac{dm_{\nu_i}}{dt} \mathbf{v}_{\nu_i} \,, \qquad (1)$$

where $\mathbf{v}_{\nu_i} = \dot{\mathbf{r}}_{\nu_i}$, m_{ν_i} is mass of appropriate point, $\mathbf{M}_P^e$ is the principal moment of the external forces, $\mathbf{M}_P^R$ is the principal moment of the reactive (jet) forces, and $\mathbf{S}_i^e$ is the sum of the angular momentum of the particles of body *i*, rejected in unit time in their translational motion with respect to the fixed system of coordinates. The angular momentum of a system of *k* bodies in coordinates system $P\xi\eta\zeta$ (Fig. 2) is defined by the following expression:

$$\begin{aligned}\mathbf{K}_P &= \sum_{i=1}^{k} \sum_{\nu_i} \mathbf{r}_{\nu_i} \times m_{\nu_i} \mathbf{v}_{\nu_i} \\ &= \sum_{i=1}^{k} \sum_{\nu_i} \left[(\mathbf{r}_0 + \boldsymbol{\rho}_{\nu_i}) \times m_{\nu_i} (\mathbf{v}_0 + \boldsymbol{\omega}_i \times \boldsymbol{\rho}_{\nu_i}) \right] \qquad (2) \\ &= \sum_{i=1}^{k} \left(\mathbf{K}_{i,O} + \mathbf{r}_O \times m_i \mathbf{v}_{C_i} + \boldsymbol{\rho}_{C_i} \times m_i \mathbf{v}_O \right),\end{aligned}$$

where

$$\mathbf{K}_{i,O} = \sum_{\nu_i} m_{\nu_i} \boldsymbol{\rho}_{\nu_i} \times \boldsymbol{\omega}_i \times \boldsymbol{\rho}_{\nu_i} \,, \qquad m_i = \sum_{\nu_i} m_{\nu_i} \,,$$
$$\mathbf{v}_{C_i} = \mathbf{v}_O + \boldsymbol{\omega}_i \times \boldsymbol{\rho}_{C_i} \,,$$

$\boldsymbol{\rho}_{C_i}$ is the radius vector of the centre of mass C_i of body *i* in the *OXYZ* system and $\boldsymbol{\omega}_i$ is the absolute angular velocity of body *i* (and coordinates system $Ox_iy_iz_i$).

In order to write the theorem of change in the angular momentum in the *OXYZ* coordinate system we need to implement some auxiliary expressions:

$$\begin{aligned}&\frac{d\boldsymbol{\rho}_{\nu_i}}{dt} = \boldsymbol{\omega}_i \times \boldsymbol{\rho}_{\nu_i} \,, \qquad \frac{d\boldsymbol{\rho}_{C_i}}{dt} = \boldsymbol{\omega}_i \times \boldsymbol{\rho}_{C_i} + \mathbf{q}_{C_i} \,, \\ &\frac{d\mathbf{v}_{C_i}}{dt} = \frac{d}{dt}\left[\mathbf{v}_O + \boldsymbol{\omega}_i \times \boldsymbol{\rho}_{C_i} \right] = \mathbf{w}_{C_i} + \boldsymbol{\omega}_i \times \mathbf{q}_{C_i} \,, \\ &\mathbf{w}_{C_i} = \mathbf{w}_O + \boldsymbol{\varepsilon}_i \times \boldsymbol{\rho}_{C_i} + \boldsymbol{\omega}_i \times \boldsymbol{\omega}_i \times \boldsymbol{\rho}_{C_i} \,, \\ &\sum_{\nu_i} \frac{dm_{\nu_i}}{dt} \left(\boldsymbol{\omega}_i \times \boldsymbol{\rho}_{\nu_i} \right) = \boldsymbol{\omega}_i \times \left[\frac{dm_i}{dt} \boldsymbol{\rho}_{C_i} + m_i \mathbf{q}_{C_i} \right],\end{aligned} \qquad (3)$$

where $\mathbf{q}_{C_i}$ is the relative velocity of the centre of mass C_i due to a change in its position with respect to the bodies, due to the variability of their masses; $\mathbf{w}_{C_i}$ is an acceleration of the point of body *i*, that currently matches with its center of mass, i.e. it is an acceleration of translation for a center of mass C_i, $\mathbf{w}_O$ is the acceleration of point *O*.

Let's prove validity of the last expression from an expression group (3):

$$\begin{aligned}\sum_{\nu_i} \frac{dm_{\nu_i}}{dt} \left(\boldsymbol{\omega}_i \times \boldsymbol{\rho}_{\nu_i} \right) &= \boldsymbol{\omega}_i \times \sum_{\nu_i} \frac{dm_{\nu_i}}{dt} \boldsymbol{\rho}_{\nu_i} \\ &= \boldsymbol{\omega}_i \times \left[\frac{d}{dt} \left(\sum_{\nu_i} m_{\nu_i} \boldsymbol{\rho}_{\nu_i} \right) - \sum_{\nu_i} m_{\nu_i} \dot{\boldsymbol{\rho}}_{\nu_i} \right] \\ &= \boldsymbol{\omega}_i \times \left[\frac{d}{dt} \left(m_i \boldsymbol{\rho}_{C_i} \right) - \sum_{\nu_i} m_{\nu_i} \left(\boldsymbol{\omega}_i \times \boldsymbol{\rho}_{\nu_i} \right) \right] \\ &= \boldsymbol{\omega}_i \times \left[\frac{dm_i}{dt} \boldsymbol{\rho}_{C_i} + m_i \dot{\boldsymbol{\rho}}_{C_i} - \boldsymbol{\omega}_i \times \sum_{\nu_i} m_{\nu_i} \boldsymbol{\rho}_{\nu_i} \right] \\ &= \boldsymbol{\omega}_i \times \left[\frac{dm_i}{dt} \boldsymbol{\rho}_{C_i} + m_i \left(\mathbf{q}_{C_i} + \boldsymbol{\omega}_i \times \boldsymbol{\rho}_{C_i} \right) - \boldsymbol{\omega}_i \times m_i \boldsymbol{\rho}_{C_i} \right] \\ &= \boldsymbol{\omega}_i \times \left[\frac{dm_i}{dt} \boldsymbol{\rho}_{C_i} + m_i \mathbf{q}_{C_i} \right].\end{aligned}$$

The relative motion of center mass with a velocity $\mathbf{q}_{C_i}$ can be illustrated with Fig. 3, which describes a burning process and corresponding geometric displacement of center of mass.

Using expressions (2) and (3) it is possible to calculate the angular momentum derivative:

$$\frac{d\mathbf{K}_P}{dt}=\sum_{i=1}^{k}\left[\frac{d\mathbf{K}_{i,O}}{dt}+\mathbf{r}_O\times\frac{dm_i}{dt}\mathbf{v}_{C_i}\right.$$
$$+\mathbf{r}_O\times m_i\left(\mathbf{w}_{C_i}+\boldsymbol{\omega}_i\times\mathbf{q}_{C_i}\right)+\mathbf{q}_{C_i}\times m_i\mathbf{v}_O \qquad (4)$$
$$\left.+\boldsymbol{\rho}_{C_i}\times\frac{dm_i}{dt}\mathbf{v}_O+\boldsymbol{\rho}_{C_i}\times m_i\mathbf{w}_O\right].$$

Let's transform the terms on the right hand side of the equation (1):

$$\mathbf{M}_P^e=\mathbf{r}_O\times\mathbf{F}^e+\mathbf{M}_O^e,\quad \mathbf{M}_P^R=\mathbf{r}_O\times\boldsymbol{\Phi}^R+\mathbf{M}_O^R,$$
$$\mathbf{S}_i^e=\mathbf{r}_O\times\frac{dm_i}{dt}\mathbf{v}_O+\mathbf{r}_O\times m_i\left(\boldsymbol{\omega}_i\times\mathbf{q}_{C_i}\right)$$
$$+\mathbf{r}_O\times\frac{dm_i}{dt}\left(\boldsymbol{\omega}_i\times\boldsymbol{\rho}_{C_i}\right)+\left[m_i\mathbf{q}_{C_i}+\frac{dm_i}{dt}\boldsymbol{\rho}_{C_i}\right]\times\mathbf{v}_O \qquad (5)$$
$$+\sum_{\nu_i}\boldsymbol{\rho}_{\nu_i}\times\frac{dm_{\nu_i}}{dt}\left(\boldsymbol{\omega}_i\times\boldsymbol{\rho}_{\nu_i}\right),$$

where $\mathbf{F}^e=\sum_{i=1}^{k}\mathbf{F}_i^e$ is resultant of system of external forces, $\boldsymbol{\Phi}^R=\sum_{i=1}^{k}\boldsymbol{\Phi}_i^R$ is resultant of reactive (jet) forces, $\mathbf{M}_O^e$, $\mathbf{M}_O^R$ are the principal moments of the external and reactive forces with respect to the point *O*.

Using expressions (4) and (5), after like terms cancellation we can rewrite a theorem (1) in the following form:

$$\sum_{i=1}^{k}\left[\frac{d\mathbf{K}_{i,O}}{dt}+\mathbf{r}_O\times m_i\mathbf{w}_{C_i}+\boldsymbol{\rho}_{C_i}\times m_i\mathbf{w}_O\right]$$
$$=\mathbf{r}_O\times\mathbf{F}^e+\mathbf{r}_O\times\boldsymbol{\Phi}^R+\mathbf{M}_O^e+\mathbf{M}_O^R \qquad (6)$$
$$+\sum_{i=1}^{k}\sum_{\nu_i}\boldsymbol{\rho}_{\nu_i}\times\frac{dm_{\nu_i}}{dt}\left(\boldsymbol{\omega}_i\times\boldsymbol{\rho}_{\nu_i}\right).$$

From the definition of center of mass and from the theorem on the motion of the center of mass of a system of variable mass [23] the following expressions must hold:

$$m_i\mathbf{w}_{C_i}=\mathbf{F}_i^e+\boldsymbol{\Phi}_i^R+\sum_{j\neq i}\mathbf{N}_{ij},\quad \mathbf{N}_{ij}=-\mathbf{N}_{ji},$$
$$\sum_{i=1}^{k}m_i\mathbf{w}_{C_i}=\mathbf{F}^e+\boldsymbol{\Phi}^R,\quad \sum_{i=1}^{k}m_i\boldsymbol{\rho}_{C_i}=m\boldsymbol{\rho}_C, \qquad (7)$$

where $m=m(t)=\sum_{i=1}^{k}m_i(t)$ is mass of the system, $\boldsymbol{\rho}_C$ is vector of center of mass *C* of the system, $\mathbf{N}_{ij}$ are internal forces of interaction between bodies.

Using expressions (6) and (7) we can write the theorem on the change in the angular momentum with respect to *OXYZ* system [28]:

$$\sum_{i=1}^{k}\frac{d\mathbf{K}_{i,O}}{dt}=\mathbf{M}_O^e+\mathbf{M}_O^R$$
$$+\sum_{i=1}^{k}\sum_{\nu_i}\boldsymbol{\rho}_{\nu_i}\times\frac{dm_{\nu_i}}{dt}\left(\boldsymbol{\omega}_i\times\boldsymbol{\rho}_{\nu_i}\right) \qquad (8)$$
$$-\boldsymbol{\rho}_C\times m\mathbf{w}_O.$$

Expression (8) corresponds to the assertion of the well-known theorem [23], taking into account the grouping of the terms according to the membership of the points of the body *i* (*i* = 1, ..., *k*).

Using the idea of a local derivative for the angular momentum vector of each body in the system of coordinates $Ox_iy_iz_i$ connected with the body, rotating with respect to *OXYZ* with angular velocity $\boldsymbol{\omega}_i$ Eq. (8) can be rewritten as follows:

$$\sum_{i=1}^{k}\left[\left(\frac{\tilde{d}\mathbf{K}_{i,O}}{dt}\right)_{Ox_iy_iz_i}+\boldsymbol{\omega}_i\times\mathbf{K}_{i,O}\right]=$$
$$\mathbf{M}_O^e+\mathbf{M}_O^R+\sum_{i=1}^{k}\sum_{\nu_i}\boldsymbol{\rho}_{\nu_i}\times\frac{dm_{\nu_i}}{dt}\left(\boldsymbol{\omega}_i\times\boldsymbol{\rho}_{\nu_i}\right)-\boldsymbol{\rho}_C\times m\mathbf{w}_O. \qquad (9)$$

The subscript outside the brackets of the local derivatives indicates a coordinate system in which they were taken. Equation (9) expresses a vector-based form of the theorem of the change in the angular momentum of bodies of variable mass with respect to the translating axes.

3. Attitude Motion of Two Variable Mass Coaxial Bodies System

We will consider the motion of a system of two bodies, where only 1st has a variable mass. Body 2 does not change its inertial and mass characteristics, calculated in the system of coordinates $Ox_2y_2z_2$ connected to the body, and, consequently, produces no reactive forces. This mechanical model can be used for research of attitude motion of dual-spin SC with operating solid-propellant rocket engine (coaxial body 1).

We will write the angular velocities and the angular momentum of the bodies in projections onto the axes of their connected systems of coordinates:

$$\boldsymbol{\omega}_i=p_i\mathbf{i}_i+q_i\mathbf{j}_i+r_i\mathbf{k}_i,\qquad \mathbf{K}_{i,O}=\mathbf{I}_i\cdot\boldsymbol{\omega}_i, \qquad (10)$$

where $\mathbf{I}_i$ are inertia tensors of body *i*; $\{\mathbf{i}_i,\mathbf{j}_i,\mathbf{k}_i\}$ are the unit vectors of the system $Ox_iy_iz_i$.

If both tensors are general then angular momentum of the bodies in projections onto the axes of their connected systems of coordinates is defined by

$$\mathbf{K}_{1,O} = A_1(t)p_1\mathbf{i}_1 + B_1(t)q_1\mathbf{j}_1 + C_1(t)r_1\mathbf{k}_1,$$
$$\mathbf{K}_{2,O} = A_2p_2\mathbf{i}_2 + B_2q_2\mathbf{j}_2 + C_2r_2\mathbf{k}_2,$$

where A_i, B_i, C_i are general moments of inertia of body i, calculated in the corresponding system of coordinates connected to the body.

The bodies of the system can only rotate with respect to one another in the direction of the common longitudinal axis, which coincides with Oz_2 (and with Oz_1). Here we will denote the angle and velocity of twisting of body 1 with respect to body 2 in the direction of the longitudinal axis Oz_2 by δ $\left(\delta = \angle(Ox_1, Ox_2)\right)$ and $\sigma = \dot{\delta}$ respectively. The angles $\{\psi, \gamma, \varphi\}$ of spatial orientation of the coaxial bodies with respect to the translating system of coordinates *OXYZ* are indicated in Fig. 4. The ratio between the angular velocities and the angular accelerations of two bodies in vector form are defined by

$$\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 + \boldsymbol{\sigma}, \qquad \boldsymbol{\varepsilon}_1 = \boldsymbol{\varepsilon}_2 + \dot{\boldsymbol{\sigma}}, \tag{11}$$

where $\boldsymbol{\sigma} = \sigma\mathbf{k}_1$ is the vector of the relative angular velocity of the bodies, which has the only projection – onto common axis of rotation Oz_2. The ratio between the components of the angular velocities for the two bodies is expressed by the following equations:

$$\begin{aligned} p_1 &= p_2\cos\delta + q_2\sin\delta, \\ q_1 &= q_2\cos\delta - p_2\sin\delta, \\ r_1 &= r_2 + \sigma. \end{aligned} \tag{12}$$

The theorem on the change in the angular momentum (9) in translating system of coordinates *OXYZ* can be rewritten in the form:

$$\begin{aligned} &\left[\left(\frac{\tilde{d}\mathbf{K}_{1,O}}{dt}\right)_{Ox_1y_1z_1} - \sum_{\nu_1}\boldsymbol{\rho}_{\nu_1}\times\frac{dm_{\nu_1}}{dt}\left(\boldsymbol{\omega}_1\times\boldsymbol{\rho}_{\nu_1}\right)\right] \\ &+\left(\frac{\tilde{d}\mathbf{K}_{2,O}}{dt}\right)_{Ox_2y_2z_2} + \sum_{i=1}^{2}\boldsymbol{\omega}_i\times\mathbf{K}_{i,O} \\ &= \mathbf{M}_O^e + \mathbf{M}_O^R - \boldsymbol{\rho}_C\times m\mathbf{w}_O, \end{aligned} \tag{13}$$

where

$$\boldsymbol{\rho}_{\nu_1} = x_{\nu_1}\mathbf{i}_1 + y_{\nu_1}\mathbf{j}_1 + z_{\nu_1}\mathbf{k}_1.$$

By projecting the expression inside the square brackets (Eq. (13)) onto the axes of the system $Ox_1y_1z_1$ and using expressions (10) we obtain:

$$\begin{aligned} \mathbf{L} &= \left[\left(\frac{\tilde{d}\mathbf{K}_{1,O}}{dt}\right)_{Ox_1y_1z_1} - \sum_{\nu_1}\boldsymbol{\rho}_{\nu_1}\times\frac{dm_{\nu_1}}{dt}\left(\boldsymbol{\omega}_1\times\boldsymbol{\rho}_{\nu_1}\right)\right] \\ &= \left(I_{1,xx}(t)\dot{p}_1 - I_{1,xy}(t)\dot{q}_1 - I_{1,xz}(t)\dot{r}_1\right)\mathbf{i}_1 \\ &\quad+\left(-I_{1,yx}(t)\dot{p}_1 + I_{1,yy}(t)\dot{q}_1 - I_{1,yz}(t)\dot{r}_1\right)\mathbf{j}_1 \\ &\quad+\left(-I_{1,zx}(t)\dot{p}_1 - I_{1,zy}(t)\dot{q}_1 + I_{1,zz}(t)\dot{r}_1\right)\mathbf{k}_1. \end{aligned} \tag{14}$$

During the simplification of equation (14) terms containing the derivatives of time-varying moment of inertia cancel out with terms following from the sum in square brackets (vector **L**). This is vividly reflected in the projection of **L** onto the connected axis Ox_1:

$$\begin{aligned} L_{x_1} &= \frac{d\left(\mathbf{K}_{1,O}\right)_{x_1}}{dt} - \sum_{\nu_1}\left[\boldsymbol{\omega}_1\rho_{\nu_1}^2 - \boldsymbol{\rho}_{\nu_1}\left(\boldsymbol{\rho}_{\nu_1}\cdot\boldsymbol{\omega}_1\right)\right]_{x_1}\dot{m}_{\nu_1} \\ &= \frac{d\left(\mathbf{K}_{1,O}\right)_{x_1}}{dt} - p_1\sum_{\nu_1}\frac{dm_{\nu_1}}{dt}\left(y_{\nu_1}^2 + z_{\nu_1}^2\right) \\ &\quad+ q_1\sum_{\nu_1}\frac{dm_{\nu_1}}{dt}x_{\nu_1}y_{\nu_1} + r_1\sum_{\nu_1}\frac{dm_{\nu_1}}{dt}x_{\nu_1}z_{\nu_1} \\ &= \frac{d}{dt}\left(I_{1,xx}(t)p_1 - I_{1,xy}(t)q_1 - I_{1,xz}(t)r_1\right) \\ &\quad- p_1\frac{dI_{1,xx}(t)}{dt} + q_1\frac{dI_{1,xy}(t)}{dt} + r_1\frac{dI_{1,xz}(t)}{dt} \\ &= I_{1,xx}(t)\dot{p}_1 - I_{1,xy}(t)\dot{q}_1 - I_{1,xz}(t)\dot{r}_1. \end{aligned}$$

If tensors of inertia remain general for each moment of time $\left(I_{1,ij}(t) \equiv 0,\ i \neq j\right)$, then vector **L** may be rewritten:

$$\mathbf{L} = A_1(t)\dot{p}_1\mathbf{i}_1 + B_1(t)\dot{q}_1\mathbf{j}_1 + C_1(t)\dot{r}_1\mathbf{k}_1. \tag{15}$$

Taking expressions (14) and (12) into account, we can write Eq. (13) in terms of projections onto the axis of $Ox_2y_2z_2$ system, connected with body 2. When changing from system $Ox_1y_1z_1$ to system $Ox_2y_2z_2$ we will use an orthogonal matrix $[\boldsymbol{\delta}]$ of rotation by angle δ. As a result we obtain:

$$\begin{aligned} &[\boldsymbol{\delta}]\left(\mathbf{L} + \boldsymbol{\omega}_1\times\mathbf{K}_{1,O}\right)_{Ox_1y_1z_1} \\ &+\left(\frac{\tilde{d}\mathbf{K}_{2,O}}{dt} + \boldsymbol{\omega}_2\times\mathbf{K}_{2,O}\right)_{Ox_2y_2z_2} \\ &= \mathbf{M}_O^e + \mathbf{M}_O^R - \boldsymbol{\rho}_C\times m\mathbf{w}_O, \end{aligned} \tag{16}$$

where

$$[\boldsymbol{\delta}]=\begin{bmatrix}\cos\delta & -\sin\delta & 0\\ \sin\delta & \cos\delta & 0\\ 0 & 0 & 1\end{bmatrix}.$$

From the theorem on the motion of the centre of mass of a system of variable mass [23] the following expressions must hold:

$$\begin{aligned}&m_1(t)\mathbf{w}_{C_1}=\mathbf{F}_1^e+\boldsymbol{\Phi}_1{}^R+\mathbf{N}_{12},\\ &m_2\mathbf{w}_{C_2}=\mathbf{F}_2^e+\mathbf{N}_{21},\\ &m_1(t)\mathbf{w}_{C_1}+m_2\mathbf{w}_{C_2}=\mathbf{F}^e+\boldsymbol{\Phi}_1{}^R,\\ &\mathbf{F}^e=\mathbf{F}_1^e+\mathbf{F}_2^e,\quad \mathbf{N}_{12}=-\mathbf{N}_{21},\end{aligned}\tag{17}$$

where $\mathbf{F}^e$ is resultant of system of external forces, $\boldsymbol{\Phi}_1{}^R$ is reactive (jet) force, $\mathbf{N}_{ij}$ are internal forces of interaction between bodies (j, k=1, 2).

The motion of center of mass of body 1 is easier to analyze as compound motion, where the motion of body 2 is translational. Considering the last remark, expressions for the acceleration of the center of mass will have the following form

$$\begin{aligned}&\mathbf{w}_{C_2}=\mathbf{w}_O+\boldsymbol{\varepsilon}_2\times\boldsymbol{\rho}_{C_2}+\boldsymbol{\omega}_2\times\boldsymbol{\omega}_2\times\boldsymbol{\rho}_{C_2},\\ &\mathbf{w}_{C_1}=\mathbf{w}^e+\mathbf{w}^r+\mathbf{w}^c,\\ &\mathbf{w}^e=\mathbf{w}_O+\boldsymbol{\varepsilon}_2\times\boldsymbol{\rho}_{C_1}+\boldsymbol{\omega}_2\times\boldsymbol{\omega}_2\times\boldsymbol{\rho}_{C_1},\\ &\mathbf{w}^r=\dot{\boldsymbol{\sigma}}\times\boldsymbol{\rho}_{C_1}+\boldsymbol{\sigma}\times\boldsymbol{\sigma}\times\boldsymbol{\rho}_{C_1},\\ &\mathbf{w}^c=2\boldsymbol{\omega}_2\times\left(\boldsymbol{\sigma}\times\boldsymbol{\rho}_{C_1}\right),\end{aligned}\tag{18}$$

where $\mathbf{w}^e$ – acceleration of translation, $\mathbf{w}^r$ – relative acceleration, $\mathbf{w}^c$ – Coriolis acceleration.

Expressions (18) imply:

$$\begin{aligned}m_1(t)\mathbf{w}_{C_1}+m_2\mathbf{w}_{C_2}=m\mathbf{w}_O+\boldsymbol{\varepsilon}_2\times m\boldsymbol{\rho}_C\\ +\boldsymbol{\omega}_2\times\boldsymbol{\omega}_2\times m\boldsymbol{\rho}_C+m_1(t)\Big(\dot{\boldsymbol{\sigma}}\times\boldsymbol{\rho}_{C_1}\\ +\boldsymbol{\sigma}\times\boldsymbol{\sigma}\times\boldsymbol{\rho}_{C_1}+2\boldsymbol{\omega}_2\times\left(\boldsymbol{\sigma}\times\boldsymbol{\rho}_{C_1}\right)\Big).\end{aligned}$$

From last relation and (17) expression for acceleration $\mathbf{w}_O$ follow:

$$\begin{aligned}\mathbf{w}_O=\frac{1}{m}\Big[\mathbf{F}^e+\boldsymbol{\Phi}_1{}^R-\boldsymbol{\varepsilon}_2\times m\boldsymbol{\rho}_C-\boldsymbol{\omega}_2\times\boldsymbol{\omega}_2\times m\boldsymbol{\rho}_C\\ -m_1(t)\Big(\dot{\boldsymbol{\sigma}}\times\boldsymbol{\rho}_{C_1}+\boldsymbol{\sigma}\times\boldsymbol{\sigma}\times\boldsymbol{\rho}_{C_1}+2\boldsymbol{\omega}_2\times\left(\boldsymbol{\sigma}\times\boldsymbol{\rho}_{C_1}\right)\Big)\Big].\end{aligned}\tag{19}$$

The $\left(-\boldsymbol{\rho}_C\times m\mathbf{w}_O\right)$ vector is represented using Eq. (19):

$$\begin{aligned}-\boldsymbol{\rho}_C\times m\mathbf{w}_O=-\boldsymbol{\rho}_C\times\Big[\mathbf{F}^e+\boldsymbol{\Phi}_1{}^R\\ -\boldsymbol{\varepsilon}_2\times m(t)\boldsymbol{\rho}_C-\boldsymbol{\omega}_2\times\boldsymbol{\omega}_2\times m(t)\boldsymbol{\rho}_C\\ -m_1(t)\Big(\dot{\boldsymbol{\sigma}}\times\boldsymbol{\rho}_{C_1}+\boldsymbol{\sigma}\times\boldsymbol{\sigma}\times\boldsymbol{\rho}_{C_1}\\ +2\boldsymbol{\omega}_2\times\left(\boldsymbol{\sigma}\times\boldsymbol{\rho}_{C_1}\right)\Big)\Big].\end{aligned}\tag{20}$$

If the change of mass of body 1, which has general tensors of inertia, is uniform along the whole volume, then tensors of inertia remain general tensors of inertia and the centre of mass of body 1 remains on a common axis of rotation Oz_2. Thus we will consider, that the body 2 also has general tensors of inertia. The following expressions will take place in this case in terms of projections onto the axes of $Ox_2y_2z_2$:

$$\begin{aligned}&\dot{\boldsymbol{\sigma}}=\left[0,0,\ddot{\delta}\right]^T,\qquad \mathbf{q}_C=\left[0,0,q_C\right]^T,\\ &\boldsymbol{\rho}_{C_1}=\left[0,0,\rho_{C_1}(t)\right]^T,\quad \boldsymbol{\rho}_C=\left[0,0,\rho_C(t)\right]^T,\\ &\left[-\boldsymbol{\rho}_C\times m\mathbf{w}_O\right]_{Ox_2y_2z_2}=\left[-\boldsymbol{\rho}_C\times\mathbf{F}^e-\boldsymbol{\rho}_C\times\boldsymbol{\Phi}_1{}^R\right]_{Ox_2y_2z_2}\\ &\qquad+m(t)\rho_C^2\Big[\left(\dot{p}_2-r_2q_2\right)\mathbf{i}_2\\ &\qquad+\left(\dot{q}_2+r_2p_2\right)\mathbf{j}_2+0\mathbf{k}_2\Big].\end{aligned}\tag{21}$$

Let's transform the moments of external and reactive (jet) forces in expression (16):

$$\begin{aligned}\mathbf{M}_O^e=\boldsymbol{\rho}_C\times\mathbf{F}^e+\mathbf{M}_C^e,\\ \mathbf{M}_O^R=\boldsymbol{\rho}_C\times\boldsymbol{\Phi}_1^R+\mathbf{M}_C^R.\end{aligned}$$

Taking the expressions (21) into account, we will write Eqs. (16) in the matrix form:

$$\begin{aligned}&[\boldsymbol{\delta}]\left\{\begin{bmatrix}A_1(t)\dot{p}_1\\ B_1(t)\dot{q}_1\\ C_1(t)\dot{r}_1\end{bmatrix}+\begin{bmatrix}\left(C_1(t)-B_1(t)\right)q_1r_1\\ \left(A_1(t)-C_1(t)\right)p_1r_1\\ \left(B_1(t)-A_1(t)\right)q_1p_1\end{bmatrix}\right\}\\ &+\left[\begin{bmatrix}A_2\dot{p}_2\\ B_2\dot{q}_2\\ C_2\dot{r}_2\end{bmatrix}+\begin{bmatrix}\left(C_2-B_2\right)q_2r_2\\ \left(A_2-C_2\right)p_2r_2\\ \left(B_2-A_2\right)q_2p_2\end{bmatrix}\right]=\\ &=\mathbf{M}_C^e+\mathbf{M}_C^R+m(t)\rho_C^2\begin{bmatrix}\dot{p}_2-r_2q_2\\ \dot{q}_2+r_2p_2\\ 0\end{bmatrix}.\end{aligned}\tag{22}$$

Components of $\{p_1,q_1,r_1\}$ in Eq. (22) must be expressed via $\{p_2,q_2,r_2\}$ using (12).

We will add an equation describing the relative motion of the bodies to the Eq. (16). A theorem on the change in the angular momentum projected onto the axis of rotation for the first body will have the following form:

$$\left(\mathbf{L}+\boldsymbol{\omega}_1\times\mathbf{K}_{1,O}\right)_{Oz_1}=M^e_{1,Oz}+M^R_z+M_\delta -\left[\boldsymbol{\rho}_{C_1}\times m_1\mathbf{w}_O\right]_{Oz_1}, \quad (23)$$

where M_δ is the moment of the internal interaction of the bodies (e.g. action of internal engine or bearing friction), $M^e_{i,Oz}$ is the moment of external forces acting only on body *i*.

If tensors of inertia of body 1 remain the general ones for every moment of time and the centre of mass of body 1 remains on a common axis of rotation Oz_2, then Eq. (23) can be rewritten in the follow form:

$$C_1(t)\dot{r}_1+\left(B_1(t)-A_1(t)\right)q_1p_1=M^e_{1,Oz}+M^R_z+M_\delta. \quad (24)$$

We will supplement the dynamic equations (16) and (23) (or their simplified analogs (22) and (24)) by the following kinematic equations (Fig.4):

$$\dot{\delta}=\sigma; \qquad \dot{\gamma}=p_2\sin\varphi+q_2\cos\varphi,$$
$$\dot{\psi}=\frac{1}{\cos\gamma}\left(p_2\cos\varphi-q_2\sin\varphi\right), \quad (25)$$
$$\dot{\varphi}=r_2-\frac{\sin\gamma}{\cos\gamma}\left(p_2\cos\varphi-q_2\sin\varphi\right).$$

Let's analyze the motion of a system of two dynamically symmetrical bodies, equations (22) and (24) will be written in the following form:

$$\begin{aligned}
&A(t)\dot{p}_2+\left(C(t)-A(t)\right)q_2r_2 +C_1(t)\,q_2\sigma=M^e_{C,x}+M^R_x,\\
&A(t)\dot{q}_2-\left(C(t)-A(t)\right)p_2r_2 -C_1(t)\,p_2\sigma=M^e_{C,y}+M^R_y,\\
&C(t)\dot{r}_2+C_1(t)\dot{\sigma}=M^e_{C,z}+M^R_z,\\
&C_1(t)\left(\dot{r}+\dot{\sigma}\right)=M_\delta+M^R_z+M^e_{1,Oz},
\end{aligned} \quad (26)$$

where

$$A(t)=A_1(t)+A_2-m(t)\rho_C^2(t), \quad C(t)=C_1(t)+C_2.$$

Systems (26) and (25) together form a complete dynamic system for the research of attitude motion of dynamically symmetrical unbalanced gyrostat with variable mass.

4. Research method of attitude motion of variable mass coaxial bodies and unbalanced gyrostat

Let's refer to a motion of coaxial bodies (unbalanced gyrostat) of variable mass under an action of dissipative and boosting external moments depending on components of angular velocities. Let the gyrostat consists of dynamically symmetrical main body (coaxial body 2) of a constant mass and a rotor (coaxial body 1) of the variable mass, which remains dynamically symmetrical during modification of a mass (Fig. 4).

The fixed point *O* coincides with an initial geometrical position of a system's center of mass. The unbalanced gyrostat has a varying relative angular velocity of rotor rotation around the main body. It is possible in connection with the existence of internal moment M_δ acting between coaxial bodies. Let's assume there is a moment of jet forces only around a longitudinal axis Oz_1 $\left(M^R_x=M^R_y=0\right)$.

Let's implement the new variables corresponding to the magnitude *G* of a vector of transversal angular velocity and angle *F* between this vector and axis Oy_2:

$$p_2=G(t)\sin F(t), \qquad q_2=G(t)\cos F(t). \quad (27)$$

Equations (26) will be rewritten in new variables as follows:

$$\begin{cases}
\dot{F}=-\dfrac{1}{A(t)}\left[\left(C(t)-A(t)\right)r_2+C_1(t)\sigma+f_F\left(G,F\right)\right],\\
\dot{G}=\dfrac{f_G\left(G,F\right)}{A(t)}, \qquad \dot{r}_2=\dfrac{M^e_{2,Oz}-M_\delta}{C_2},\\
\dot{\sigma}=\dfrac{C(t)M_\delta}{C_1(t)C_2}+\dfrac{M^R_z+M^e_{1,Oz}}{C_1(t)}-\dfrac{M^e_{2,Oz}}{C_2}.
\end{cases} \quad (28)$$

In equations (28) the following disturbing functions describing exposures take place:

$$f_G\left(G,F\right)=\left(M^e_{C,x}\sin F+M^e_{C,y}\cos F\right),$$
$$f_F\left(G,F\right)=\frac{1}{G}\left(M^e_{C,x}\cos F-M^e_{C,y}\sin F\right).$$

We will consider a case when the module of a transversal angular velocity of main body is small in comparison to relative longitudinal rotation rate of the rotor:

$$\varepsilon=\sqrt{p_2^2+q_2^2}\Big/\left|\sigma\right|<<1. \quad (29)$$

From spherical geometry the formula for a nutation angel θ (an angle between axes *OZ* and Oz_2) follow

$$\cos\theta=\cos\psi\cos\gamma.$$

We will assume angles γ and ψ to be small $\left(\gamma=O(\varepsilon),\ \psi=O(\varepsilon)\right)$. Then the nutation angle will be defined by the following approximated formula:

$$\theta^2\cong\gamma^2+\psi^2. \quad (30)$$

Using the expressions (22) and kinematic equations (20) we can write (second order infinitesimal terms are omitted):

$$\dot{\gamma} \cong G\cos\Phi(t), \quad \dot{\psi} \cong G\sin\Phi(t),$$
$$\dot{\varphi} \cong r_2, \quad \dot{\delta} = \sigma, \quad \Phi(t) = F(t) - \varphi(t). \tag{31}$$

Function $\Phi(t)$ is a phase of spatial oscillations.

Precession motion of the gyrostat with small nutation angles is obviously described by a phase space of variables $\{\gamma, \psi\}$. The phase trajectory in this space completely characterizes motion of the longitudinal axis Oz_2 (an apex of the longitudinal axis). Therefore our further researches will be connected to the analysis of this phase space and chances of behaviors of phase curves in this space.

We can develop a special qualitative method of the analysis of a phase space. Main idea of the method is the evaluation of a phase trajectory curvature in the phase plane $\{\gamma, \psi\}$.

On the indicated plane the phase point will have following components of a velocity and acceleration:

$$V_\gamma = \dot{\gamma}, \; V_\psi = \dot{\psi}, \; W_\gamma = \ddot{\gamma}, \; W_\psi = \ddot{\psi}.$$

With the help of expressions (26) the curvature of a phase trajectory (k) is evaluated as follows:

$$k^2 = \left(\ddot{\gamma}\dot{\psi} - \ddot{\psi}\dot{\gamma}\right)^2 \Big/ \left(\dot{\gamma}^2 + \dot{\psi}^2\right)^3 = \dot{\Phi}^2 / G^2. \tag{32}$$

If curvature magnitude increase, there will be a motion on a twisted spiral trajectory similar to a steady focal point (Fig. 5, case “a”) and if decreases - on untwisted. On twisted spiral trajectory motion condition can be noted as:

$$|k| \uparrow \Rightarrow k\dot{k} > 0 \;\Rightarrow\; \dot{\Phi}\ddot{\Phi}G - \dot{G}\dot{\Phi}^2 > 0. \tag{33}$$

For the analysis of the condition realization it is necessary to study a disposition of zero points (roots) of a following function:

$$P(t) = \dot{\Phi}\ddot{\Phi}G - \dot{G}\dot{\Phi}^2. \tag{34}$$

Function (34) will be defined as a function of phase trajectory evolutions.

Different qualitative cases of phase trajectory behaviors are possible depending on function $P(t)$ zero points of (Fig.5). In the first case (Fig. 5, case “a”) the function is positive and has no zero on a considered slice of time $t \in [0, T]$, thus the phase trajectory is spirally twisting. In the second case (Fig. 5, case “b”) there exists one zero point and there is one modification in a monotonicity of the trajectory curvature. The Cornu spiral, also known as clothoid, take place in case “b”. The third case (Fig. 5, case “c”) represents a number of zero points and the trajectory has alternation of untwisted and twisted segments of motion; also there are some points of self-intersection.

5. Research of attitude motion.
5.1. Example 1.

As an first example we will refer to a motion of coaxial bodies of variable mass under the influence of constant internal moment ($M_\delta = const$) and constant moment of jet forces ($M_z^R = const$). The analysis of a phase space is conducted using a developed method of curvature evaluation.

We will suppose that the mass and moments of inertia are linear functions of time:

$$m_1(t) = m_r - kt,$$
$$A_1(t) = \alpha m_1(t), \quad A_2 = A_m = const, \tag{35}$$
$$C_1(t) = \beta m_1(t), \quad C_2 = C_m = const,$$

where m_r is initial mass of rotor, k is rate of mass change, and α, β are constants.

Dependencies (35) are valid for a dual-spin SC when one of coaxial bodies is a solid-propellant rocket engine with packed and roll shaped grains. A linear law of mass change provides a constant thrust. In the rocket engines of a described type the grain usually burns uniformly over the whole volume, the grain density changes uniformly as well (Fig.3-c). The center of mass of an engine-body will show no displacement relative to the body, therefore $\rho_{C_1} = l_r = const$. Center of mass of the main body does not move as well, because the body doesn’t change its mass, therefore $\rho_{C_2} = l_m = const$. Let’s mark that constants l_r and l_m are in fact the distances between the bodies’ center of masses and the point O (Fig.4): $l_n = OC_2$, $l_r = OC_1$. For example, if body1 is solid cylinder, then the constants α and β are

$$\alpha = H^2/12 + R^2/4 + l_r^2, \;\; \beta = R^2/2,$$

where H is the height of a cylinder and R is the radius.

Magnitude $\rho_C(t)$ will be defined by a linear-fractional form:

$$\rho_C(t) = \frac{l_m m_2 + l_r m_1(t)}{m_2 + m_1(t)}. \tag{36}$$

At t=0 the system’s center of mass C and the point O are matching, therefore

$$\rho_C(0) = 0, \quad l_r m_1(0) = -l_m m_2,$$
$$l_m < 0, \quad l_r > 0.$$

On base (36) and (35) we will write time-dependences for $A(t)$ and $C(t)$:

$$A(t) = A - at - \frac{k^2 l_r^2 t^2}{m - kt},$$
$$C(t) = C - ct, \tag{37}$$

where

$$A = A_r + A_m, \quad A_r = A_1(0), \quad a = \alpha m_1(0),$$
$$C = C_r + C_m, \quad C_r = C_1(0), \quad c = \beta m_1(0).$$

In a considered case equations (28) will obtain the following form:

$$\dot{G} = 0,$$
$$\dot{F} = -\frac{1}{A(t)}\left[\left(C(t) - A(t)\right) r + (C_r - ct)\sigma\right], \tag{38}$$
$$\dot{r}_2 = -\frac{M_\delta}{C_m}, \qquad \dot{\sigma} = \frac{C(t) M_\delta}{(C_r - ct) C_m} + \frac{M_z^R}{C_r - ct}.$$

Analytical solutions for angular velocity $r_2(t)$ and $\sigma\left(t\right)$ are derived from equations (38):

$$r_2 = r_0 - \frac{M_\delta}{C_m} t, \quad \sigma = \sigma_0 + s_1 t + s_2 \ln\left(1 - c_1 t\right),$$
$$s_1 = M_\delta / C_m, \quad s_2 = -\frac{1}{c}\left(M_\delta + M_z^R\right), \; c_1 = c / C_r. \tag{39}$$

Kinematic equations (31) can be used to receive a solution for the angle φ:

$$\varphi = \varphi_0 + r_0 t - \frac{M_\delta}{2C_m} t^2.$$

Expression for a time derivative of a spatial oscillations phase $\dot{\Phi}$ can be obtained using (31), (39) and (38):

$$\dot{\Phi} = \frac{1}{A(t)}\Big[\left(C(t) - A(t)\right)\left(s_1 t - r_0\right)$$
$$- (C_r - ct)\left(\sigma_0 + s_1 t + s_2 \ln\left(1 - c_1 t\right)\right)\Big] \tag{40}$$
$$+ s_1 t - r_0.$$

Formula (40) make it possible to receive an explicit expansion for evolution function (34), but this expression is difficult to analyze. It is reasonable to expand this expression into a series:

$$P(t) = \dot{\Phi}\ddot{\Phi} = \sum_{i=0}^{\infty} f_i t^i. \tag{41}$$

Writing formula (41) on the basis of (34) we don't take into account a constant multiplier G_0.

Further we will investigate the simplest case when the expansion for $P(t)$ has only linear part (other terms of expansion are not taken into account). On the basis of expressions (40) we can get a polynomial of the first degree for the phase trajectory evolutions function (34):

$$P(t) \cong f_0 + f_1 t, \tag{42}$$

where

$$f_0 = \frac{C_r}{A^3}\left[\left(aC_r - cA\right)\sigma_0^2 + A M_z^R \sigma_0\right],$$

$$f_1 = \frac{3a^2 C_r^2 \sigma_0^2}{A^4} + \frac{C_r \sigma_0}{A^3}\Bigg[4aM_z^R + 2\sigma_0\left(\frac{C_r k^2 l_r^2}{m} - 2ca\right)\Bigg] + \frac{1}{A^2}\left[c^2\sigma_0^2 - c\sigma_0\left(M_\delta + 3M_z^R\right) + \left(M_z^R\right)^2\right].$$

There is a unique zero point of function $P(t)$: $t_1 = -f_0 / f_1$. For implementation of a condition (33) of twisted spiral motion it is necessary for the polynomial to be steady ($t_1 < 0$) and positive for all $t \geq 0$. It is possible only in case the following conditions fulfills:

$$f_0 > 0, \quad f_1 > 0. \tag{43}$$

We will consider a case when following contingencies are correct:

$$r_0 = 0, \quad \sigma_0 > 0, \; M_z^R > 0.$$

In this case value f_0 will be positive if following condition is true:

$$c/C_r > a/A, \qquad \sigma_0\left(cA - aC_r\right) < AM_z^R. \tag{44}$$

In order f_1 to be also positive the following conditions must be satisfied:

$$\sigma_0\left(c - \frac{C_r k^2 l_r^2}{2am}\right) < M_z^R,$$
$$3M_z^R - \frac{\left(M_z^R\right)^2}{c\sigma_0} < c\sigma_0 - M_\delta. \tag{45}$$

Also $f_1 > 0$ if following conditions hold true:

$$C_r k^2 l_r^2 > 2acm, \quad M_\delta + 3M_z^R < 0. \tag{46}$$

Figure 6 illustrates the results of evolution function and appropriate phase trajectories numeric calculations.

Figures Fig.6-a and Fig.6-b demonstrate the situation when (44) and (45) are satisfied, fig.6-c show the opposite case. Point indicates the beginning of phase trajectories. In case “a” evolution function has two roots and phase trajectory has three evolutions: twisting-untwisting-twisting. In case “b” evolution function has no root and single evolution of phase trajectory takes place – this evolution is spiral twisting. System parameters and initial conditions for obtained solutions are listed in table 1.

Table 1
Value of the system parameters for figure 6

Quantity	a	b	c
M_δ, N·m	1	-10	200
M^R_z, N·m	15	10	0.35
σ_0, radian/sec	10	1	16
G_0, radian/sec	0.2	0.2	0.2
A_m, kg·m^2	2.5	2.5	2.5
A_r, kg·m^2	2.5	1.5	2
C_m, kg·m^2	1	1	1
C_r, kg·m^2	1.5	1.5	2
a, kg·m^2/sec	0.08	0.05	0.1
c, kg·m^2/sec	0.08	0.08	0.08
l_r, m	0.5	0.4	0.6
k, kg/sec	1	1	1.2
$m_1(0)$, kg	35	25	45
m_2, kg	35	35	35

Conditions (44) and (45) can be used for the synthesis of dual-spin spacecraft parameters. In order to enhance the accuracy of SC’s longitudinal axis positioning it is necessary to realize precession motion with a decreasing nutation angle. This motion is realized when the conditions (44) and (45) are satisfied.

For realization of more accurate researches, certainly, it is necessary to take higher degrees polynomials $P(t)$ (34) into account. However it was shown that an implemented analysis provides an adequate description of the precessional motion evolutions of variable mass coaxial bodies.

Examined above case of investigation does not take into account many important aspects of variable mass coaxial bodies motion. However the introduced example has illustrated the approach to a research of non-autonomous dynamical systems of indicated type.

5.2. Example 2.

Let’s refer to the other mode of motion with the following external and internal dissipative force moments:

$$M^e_{C,x} = -\nu p, \quad M^e_{C,y} = -\nu q, \qquad (47)$$
$$M^e_{2,Oz} = -\lambda r, \quad M^e_{1,Oz} = -\mu(r+\sigma).$$

Constants ν, μ, λ describe the influence of resisting substance on a gyrostat and the dissipation of energy.

Let the mass and inertial parameters be described by polynomial functions of time:

$$A(t) = A_2 + A_1(t) - m(t)\rho_C^2(t) = \sum_{i=0}^{n} a_i t^i, \qquad (48)$$
$$C_1(t) = \sum_{i=0}^{m} c_i t^i,$$

Dynamical equations of motion will have the following form (26):

$$A(t)\dot{p}_2 + \left(C(t) - A(t)\right) q_2 r_2 + C_1(t) q_2 \sigma = -\nu p_2,$$
$$A(t)\dot{q}_2 - \left(C(t) - A(t)\right) p_2 r_2 - C_1(t) p_2 \sigma = -\nu q_2, \qquad (49)$$
$$C(t)\dot{r}_2 + C_1(t)\dot{\sigma} = M_z^R - \lambda r_2 - \mu(r_2 + \sigma),$$
$$C_1(t)\left(\dot{r}_2 + \dot{\sigma}\right) = M_\delta + M_z^R - \mu(r_2 + \sigma),$$

where $C(t) = C_2 + C_1(t)$.

The last equation in (49) provides an equation and a general solution for an absolute longitudinal angular velocity of rotor Ω:

$$\dot{\Omega} = \frac{1}{C_1(t)}\left(M_\delta + M_z^R - \mu\Omega\right),$$
$$\Omega(t) = \frac{1}{\mu}\left(M_\delta + M_z^R\right) \qquad (50)$$
$$- \frac{1}{\mu}\left(M_\delta + M_z^R - \mu\Omega_0\right)\exp\left[-\mu J_C(t)\right],$$

where

$$\Omega = r_2 + \sigma, \quad J_C(t) = \int_0^t \frac{dt}{C_1(t)}. \qquad (51)$$

Using expressions (50) we can notice that the third equation in (49) gives an equation and a general solution for a longitudinal angular velocity of the main body:

$$C_2\dot{r}_2 = -M_\delta - \lambda r_2,$$
$$r_2(t) = \left(r_0 + \frac{M_\delta}{\lambda}\right)\exp\left[\frac{-\lambda t}{C_2}\right] - \frac{M_\delta}{\lambda}. \qquad (52)$$

In the case involved disturbing functions will have the form:

$$f_G(G,F) = -\nu G, \quad f_F(G,F) = 0, \qquad (53)$$

thereby first two equations can be rewritten the following way:

$$\dot{G} = -\frac{\nu G}{A(t)},$$
$$\dot{F} = -\frac{1}{A(t)}\left[C_2 r_2 + C_1(t)\Omega - A(t) r_2\right]. \quad (54)$$

This implies a solution for the amplitude of the transverse angular velocity:

$$G(t) = G_0 \exp\left[-\nu J_A(t)\right], \quad (55)$$

where

$$J_A(t) = \int_0^t \frac{1}{A(t)}, \quad G_0 > 0. \quad (56)$$

The value of integrals contained in expressions (50) and (55) can be calculated analytically:

$$J_A(t) = \sum_{i=1}^{n} \frac{\ln|t-\alpha_i|}{\dot{A}(\alpha_i)}\Bigg|_0^t,$$
$$J_C(t) = \int_0^t \frac{dt}{C_1(t)} = \sum_{i=1}^{m} \frac{\ln|t-\beta_i|}{\dot{C}_1(\beta_i)}\Bigg|_0^t, \quad (57)$$

where α_i, β_i – are the roots of $A(t)$, $C_1(t)$ polynomials, which contain no real roots within a considered interval $t \in [0, T]$, because the moments of inertia are strictly positive quantities. Let's mark that formulas can be checked by differentiation.

Expressions (50), (52) and (55) provide final analytic solutions for the longitudinal angular velocities and the amplitude of transverse angular velocity of the gyrostat. Using the first equation in (54) we can obtain an expression for the evolution function:

$$P(t) = G\left(\frac{1}{2}\frac{d\dot{\Phi}^2}{dt} + \frac{\nu\dot{\Phi}^2}{A(t)}\right). \quad (58)$$

Let's consider a case when there are no moments of internal interaction and jet forces and the main body has no initial longitudinal angular velocity [26 – 29]:

$$M_\delta = M_z^R = 0, \quad r_0 = 0. \quad (59)$$

Let's perform some supplementary transformations of quantities presented in expression (58). Replacing derivatives $\dot{\Omega}, \dot{\Phi}$ with corresponding right hand sides of equations (54), (50) and considering that $r(t) \equiv 0$, we can write the following expressions:

$$\dot{\Phi}^2 = \frac{C_1^2(t)}{A^2(t)}\Omega^2,$$
$$\frac{d\dot{\Phi}^2}{dt} = \frac{2\Omega^2}{A^3}\left(\dot{C}_1 C_1 A - \dot{A} C_1^2 - \mu C_1 A\right). \quad (60)$$

Now function (58) will have the following form:

$$P(t) = \frac{G\Omega^2 C_1}{A^3}\left(\dot{C}_1 A - \dot{A} C_1 - \mu A + \nu C_1\right). \quad (61)$$

Solutions (55) and (50) imply that $G > 0$, $\Omega \neq 0$, therefore a multiplier, placed before brakets in (61), is strictly positive over the whole interval $t \in [0, T]$. Considering the last remark, function $P(t)$ may be replaced by the following expression:

$$P(t) = \dot{C}_1 A - \dot{A} C_1 - \mu A + \nu C_1. \quad (62)$$

Expressions (62) and (48) imply, that function $P(t)$ – is a polynomial with a degree of $N=m+n-1$, that is why a theoretically valid number of phase trajectory evolutions can not exceed $N+1$.

Using function (62) we can obtain the following limitations for the moment of inertia functions, providing the twisted-in phase trajectory and therefore the decreasing amplitude of nutation angle:

$$C(t)/A(t) > \dot{C}(t)/\dot{A}(t), \quad C(t)/A(t) > \mu/\nu. \quad (63)$$

This limitations particularly result in the condition for the linear functions of moments of inertia in the absence of dissipative moments ($\nu=\mu=0$), given in articles [27, 28]:

$$a/A > c/C_r.$$

In general case, when constraints (59) are not satisfied, conditions similar (63) have not been found. It is possible in this case to receive numerical results for the evolution function and phase trajectory (Fig. 7). The following parameters and polynomial time-dependences of inertia moments have been used for computation:

$$A(t) = -0.0001t^4 + 0.0064t^3$$
$$-0.1053t^2 + 0.4491t + 3,$$
$$C(t) = 0.0001t^4 - 0.0027t^3$$
$$+0.0344t^2 - 0.1997t + 1.92,$$
$$C_2 = 1.5 \text{ (kg}\times\text{m}^2\text{)}; \quad T = 24 \text{ (sec)};$$
$$\nu = 0.02, \ \mu = 0.01, \ \lambda = 0.03 \ \text{(kg}\times\text{m}^2\text{/sec)};$$
$$M_\delta = -11, \ M_z^R = -21 \ \text{(kg}\times\text{m}^2\text{/sec}^2\text{)};$$
$$G_0 = 0.1, \ r_0 = 15, \ \Omega_0 = 10 \text{ (rad/sec)}.$$

It can be noticed (Fig. 7) that function $P(t)$ has five roots on examined time-interval and consequently phase trajectory has six evolutions.

5.3. Example 3.

Omitting the solution details we consider the numerical simulation of previous example when inertia moments are simplest harmonic function:

$$A(t) = a_1 \sin \chi t + a_0, \quad a_0 > a_1 > 0,$$
$$C_1(t) = c_1 \cos \chi t + c_0, \quad c_0 > c_1 > 0. \qquad (64)$$

For this case solutions (50) and (55) remain valid, but expressions for integrals (51) and (56) take on a value:

$$J_C(t) = \frac{2}{\chi\sqrt{c_0^2 - c_1^2}} \left[\arctan\left(\frac{(c_0 - c_1)\tan[\chi t/2]}{\sqrt{c_0^2 - c_1^2}} \right) + \pi\left(\left(\frac{\chi t}{2} + \frac{\pi}{2} \right) \text{ div } \pi \right) \right],$$
$$J_A(t) = \frac{2}{\chi\sqrt{a_0^2 - a_1^2}} \left[\arctan\left(\frac{a_0 \tan[\chi t/2] + a_1}{\sqrt{a_0^2 - a_1^2}} \right) + \pi\left(\left(\frac{\chi t}{2} + \frac{\pi}{2} \right) \text{ div } \pi \right) \right], \qquad (65)$$

where operation (x div y) corresponds to evaluation of integer part of division x/y.

Evolution function take on form:

$$P(t) = G(t)\left[\frac{K_z^2(t)}{A^3(t)}\left(\nu - \dot{A}\right) + \frac{K_z(t)}{A^2(t)}\left(\left[\dot{C} - \mu\right]\Omega(t) + M_z^R - \lambda r_2(t) \right) \right], \qquad (66)$$

where

$$K_z(t) = C_2 r_2 + C(t)\Omega.$$

Though quite simple analytical description of evolution function (66), the phase trajectory behavior in considered case can be complex. Phase trajectory can be regular (Fig.8-a), or can demonstrate unpredictable forms, which typical in chaotic dynamics (Fig.8-b, c).

On Fig.8 three cases of phase trajectories are shown. These trajectories are calculated for parameters from table 2; constants for inertia moments dependences (64) are equal $a_0=c_0=2$, $a_1=c_1=1$ (kg×m^2), $\chi = 3$ (1/sec).

Case “a” correspond to motion without reactive and internal forces moments $\left(M_\delta = M_z^R = 0\right)$. In this case quasi-periodic evolution function (with slow damped amplitude) take place and phase trajectory is also quasi-periodic and regular.

In presence reactive and internal forces moments (cases “b” and “c”) evolution function become nonperiodic with complex changing amplitude rate. In these cases phase trajectories become nonregular and similar to chaotic. Cases “b” and “c” correspond to a positive $(\nu, \mu, \lambda > 0)$ and negative $(\nu, \mu, \lambda < 0)$ dissipation.

It is necessary to note, what all calculations were conduct in MAPLE 11 [31] with use of numerical solution of stiff initial value problem (absolute error tolerance is equal 0.0001).

Table 2
Value of the system parameters for figure 8

Quantity	a	b	c
M_δ, N·m	0	0.1	0.5
M^R_z, N·m	0	0.1	0.03
r_0, radian/sec	15	-1	1
Ω_0, radian/sec	-18	-10	-10
G_0, radian/sec	0.01	0.01	0.01
C_2, kg·m^2	2.5	2.5	2.5
ν, kg×m^2/sec	0.00001	0.001	-0.001
μ, kg×m^2/sec	0.00001	0.001	-0.002
λ, kg×m^2/sec	0.00001	0.001	-0.003
T, sec	50	470	500

6. Conclusion

The article described a research of the phase space of non-autonomous dynamical system of coaxial bodies of variable mass using a new method of phase trajectory curvature analysis.

Developed method allows to estimate the phase trajectory form.

System motion can be both simple regular and very complicated nonregular (chaotic).

Regular motions are realized at evolution functions with finite number of roots (polynomial) or at periodic evolution functions. Complicated nonregular motions arise at nonperiodic alternating evolution functions with infinite number of roots.

Results of the research have an important applied value for the problems of space flight mechanics and especially for coaxial spacecrafts.

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Legends for figures

Fig.1. Scattering of thrust and transfer orbit

Fig.2. Coaxial bodies system and coordinates system

Fig. 3. Change of center of mass position with respect to the bodies, due to the variability of their masses (cases of body burn): a – right to left burn, b – left to right burn, c – burn with uniform reduction of body density

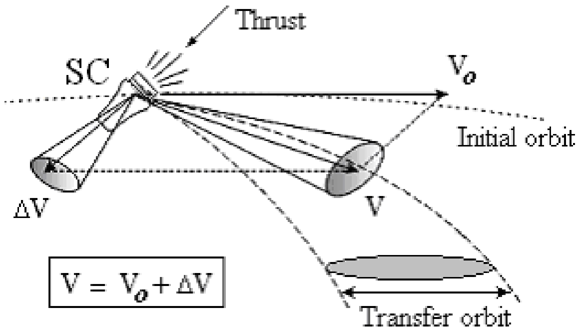
Fig.4. Two coaxial bodies system, coordinates systems and Euler’s type angels

Fig. 5. Cases of phase trajectory behaviors

Fig.6. Evolution function and cases of phase trajectory evolutions depends on conditions (44) and (45) fulfillment: in cases “a” and “b” conditions are satisfied and first evolution of phase trajectory is spiral twisting; in case “c” condition are unsatisfied and first evolution is untwisting.

Fig.7. Numerical simulation results for the evolution function and phase trajectory

Fig.8. Evolution functions and phase trajectories in case with harmonic inertia moments



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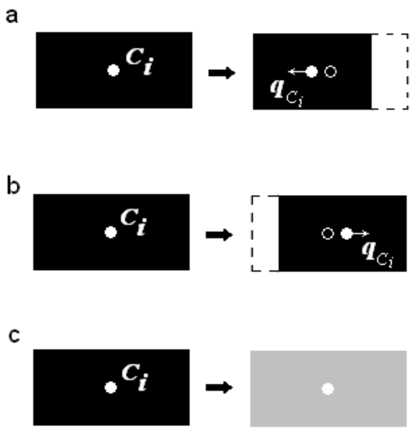

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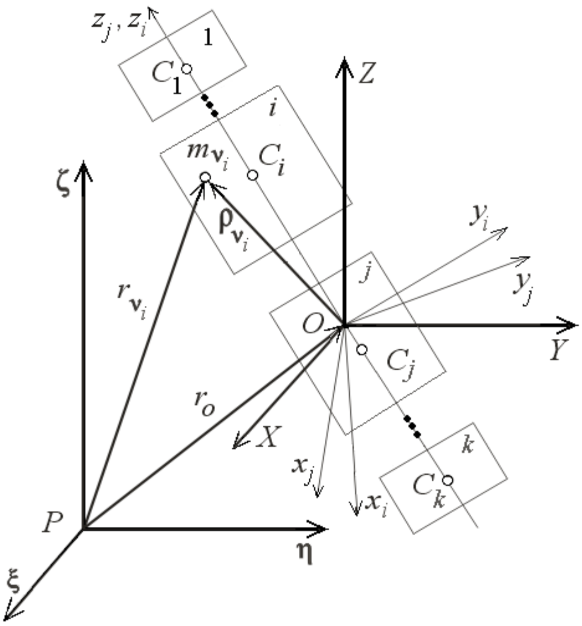

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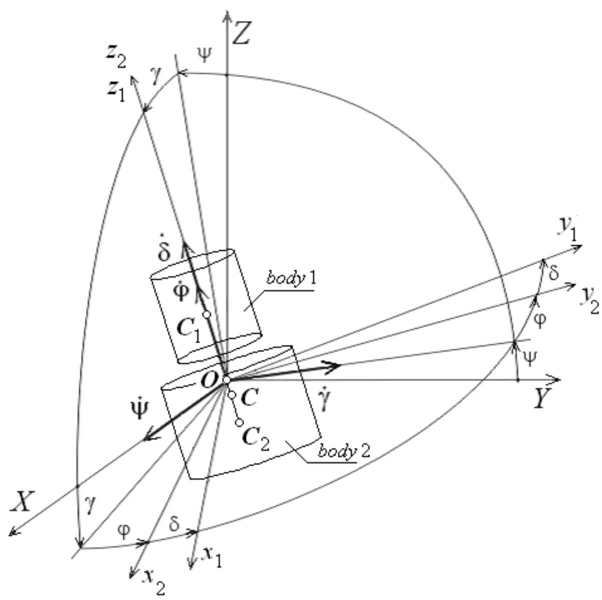

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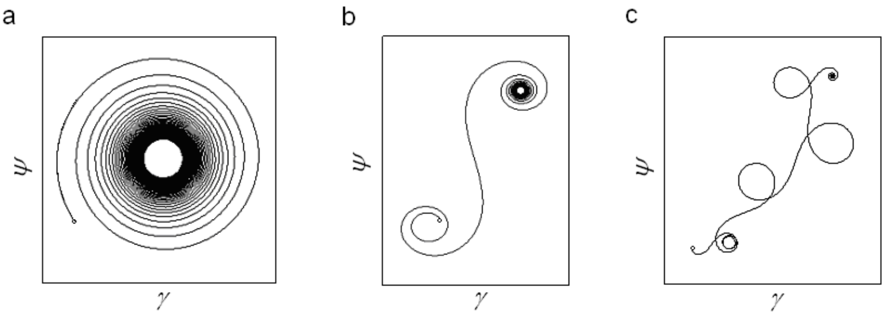
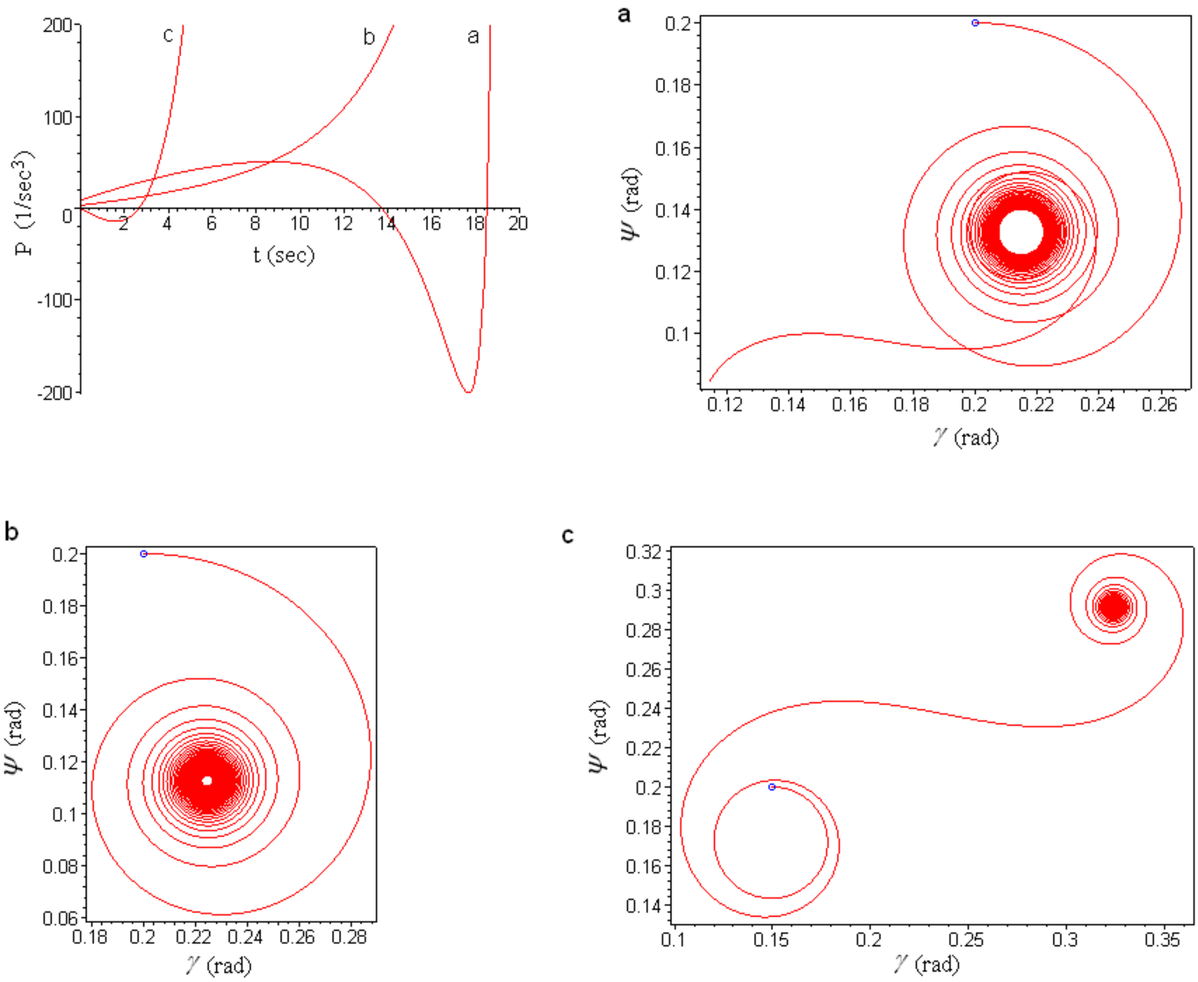


Fig. 5. Cases of phase trajectory behaviors



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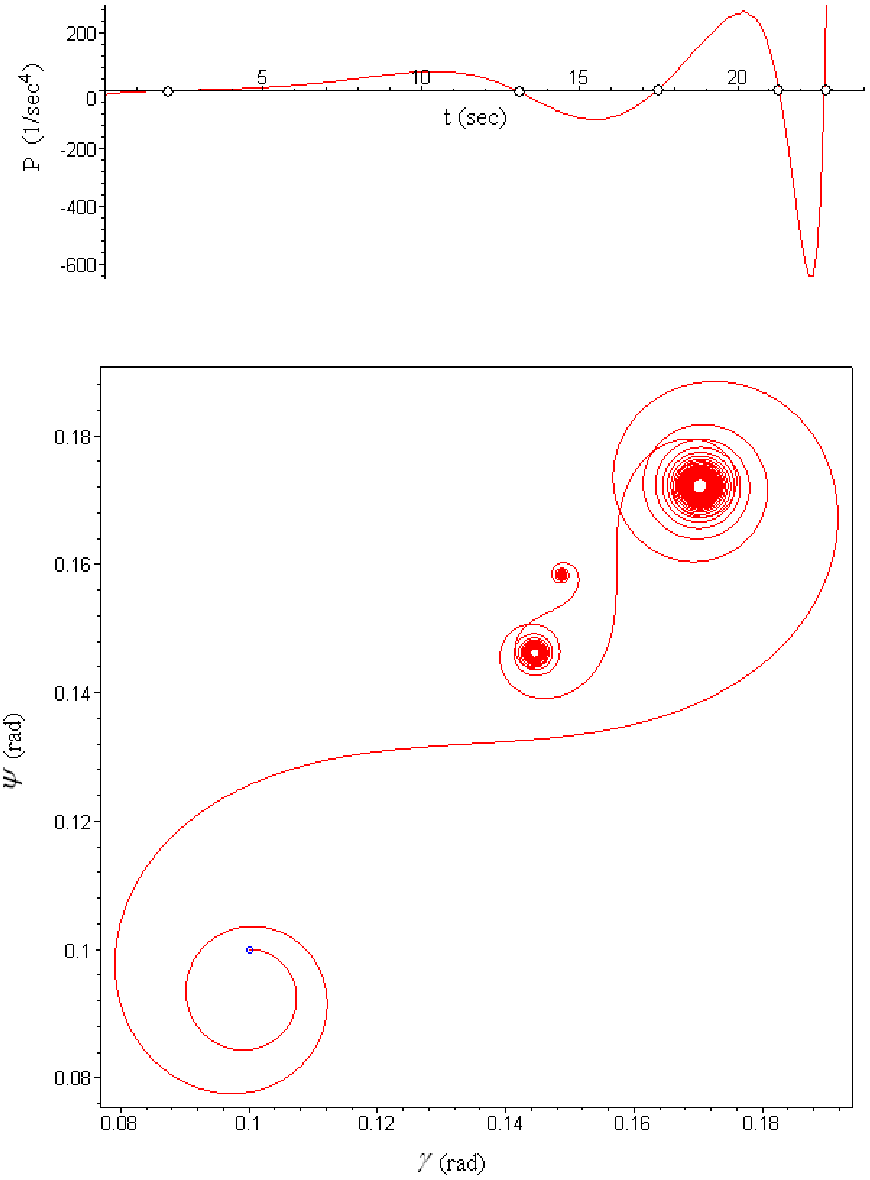

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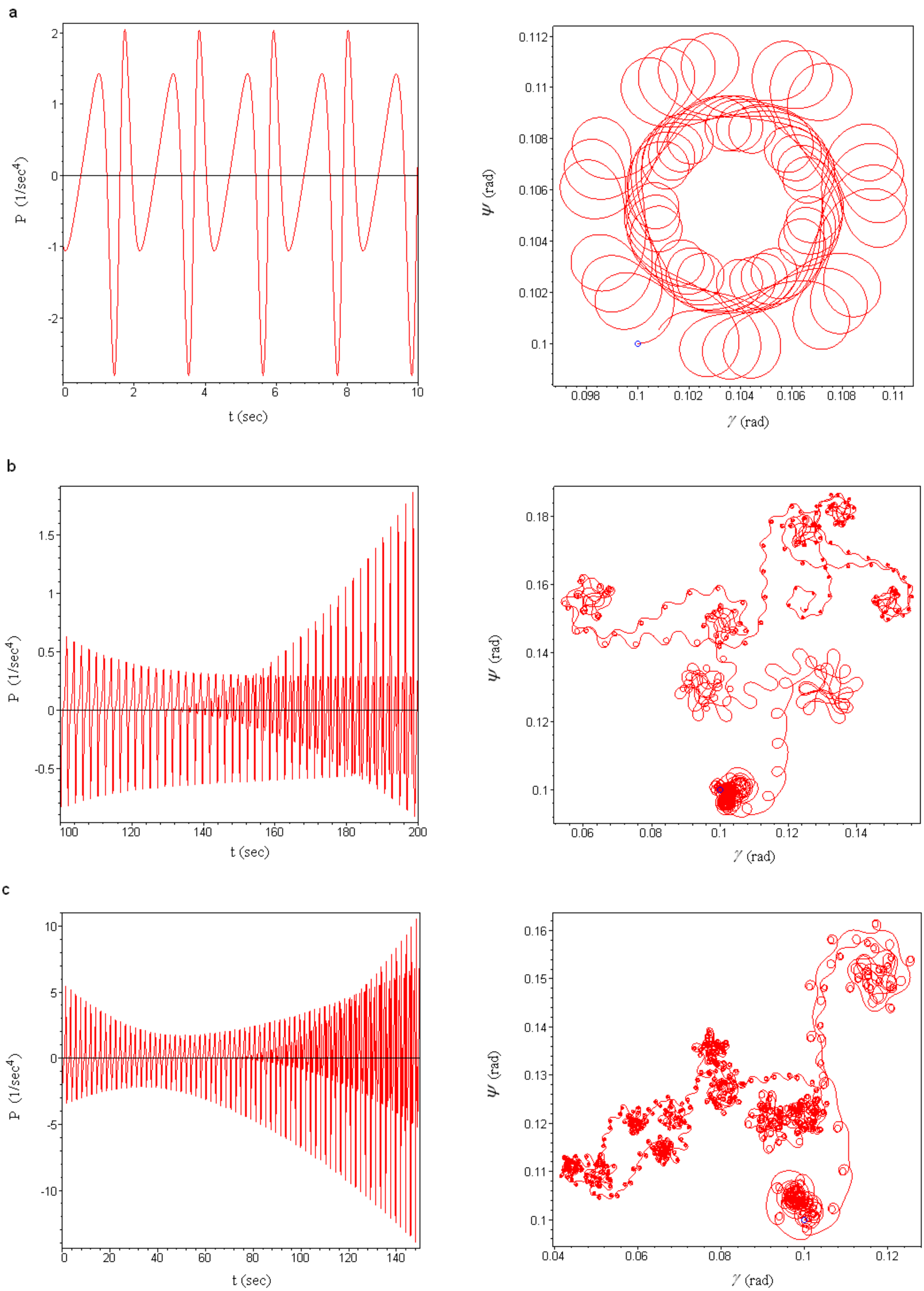

Fig.8. Evolution functions and phase trajectories in case with harmonic inertia moments