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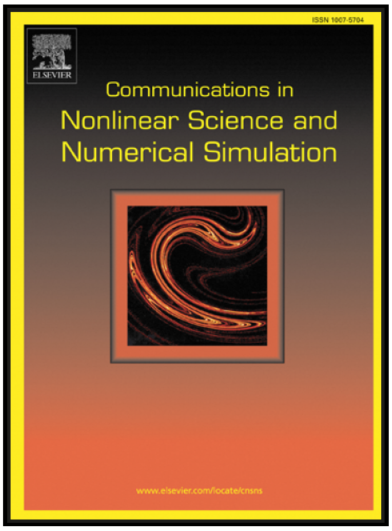

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Highlights

- The case of gyrostats angular motion in the central gravitational field is investigated at the condition of the collinearity of the angular momentum and the gravity field gradient.
- The case is closely connected with V.A. Stekloff's case of the rigid body motion and can be considered as its partial generalization on the gyrostat motion.
- The analytical solution for the gyrostat/gyrostat-satellite angular motion parameters is obtained in terms of elliptic functions.
- The possibility of chaotization phenomena in the gyrostat-satellite angular motion are investigated.

Regimes of regular and chaotic motion of gyrostats in the central gravity field

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Abstract. In this article, the partial case of gyrostats angular motion in the central gravitational field is investigated at the condition of the collinearity of the angular momentum and the gravity field gradient. This case is closely connected with V.A. Stekloff's case of the rigid body motion and can be characterized as its partial generalization on the gyrostat motion. Considered dynamical circumstances can be realized in "conical precessions" regimes of rotating prolate gyrostat-satellites at the stabilization by the gravitational way along the local vertical direction. The corresponding analytical solution for the gyrostat/gyrostat-satellite angular motion parameters is obtained in terms of elliptic functions. In addition, the possibility of chaotization phenomena in the gyrostat-satellite angular motion are investigated.

Key-Words: - Gyrostat; Satellite; Dual-Spin Spacecraft; Rigid Body Dynamics; Explicit Exact Solutions; Elliptical Integrals; Jacobi Elliptic Functions; Stekloff case; Serret-Andoyer-Deprit variables; Chaotic dynamics

Introduction

The angular motion of gyrostats and similar rigid bodies systems under the action of internal and external forces and torques remains one of the main research theme of the classical mechanics, and also has the bright spectrum of applications, including the space flight dynamics and, especially, the problem of satellites attitude dynamics. The angular motion of gyrostats in the central gravity field is an important part of the mentioned scientific problem. The task of the motion of rigid body and gyrostats in the central gravity field was considered in different formulations in many research works. Classical and new solutions linked with the task of the angular motion of rigid bodies and gyrostats in the gravity field are presented, e.g., in [1-9]. Important cases of the analysis of the different regimes and evolutions of rigid bodies motion under the gravity force, including the synthesis of the stability conditions can be found, for example, in [10-28], and, herewith the special questions of an irregular dynamics of gyrostats are considered in [29-33].

The works [5-8] should be underlined separately because they are linked with the considering task due to the ascertained dynamical analogy of the rigid body motion in a liquid with its motion in the central gravity field. This dynamical analogy was found by V.A. Stekloff [5], and generalized by P.V. Kharlamov [7] on the case of motion of gyrostats. This dynamical analogy allowed to use the solutions [6] for describing the rigid bodies motion in the central gravitational field, that was presented e.g. in [1, 7].

In this article, the partial case of the gyrostat angular motion in the weak central gravitational field is investigated at the condition of the collinearity of the angular momentum and the gravity field gradient. At such conditions it is possible to approximately consider the vector of the gyrostat/gyrostat-satellite angular momentum as the constant vector in the inertial space. Then the necessary quantity of first integrals can be found, and the analytical solution for the gyrostat/gyrostat-satellite angular motion parameters can be obtained in terms of elliptic functions. This case, as it will be shown, is not only closely connected with the V.A. Stekloff case, but can be characterized as the partial generalization of the V.A. Stekloff case on the gyrostat motion - this circumstance, as well as new analytical solutions, defines the fundamental side of the work.

From the another point of view, the considered case describes the attitude dynamics of an axial gyrostat-satellite in the central gravity field at the realizations of "conical precessions" regimes [10], when it is stabilized by the gravitational way. This defines the applied side of the work. The indicated conical precession is one of the most useful case of the attitude stabilization of prolate spacecraft on the orbit, when its longitudinal axis fulfills the precessional motion around the local “vertical direction” connecting the gravity center with the spacecraft center of mass.

In addition, the possibility of chaotization phenomena in the satellite angular motion close the conical precession are investigated in the article – this also can be indicated as the important result. It is known, that the dynamical chaos is the important phenomenon studied by the modern science in the fundamental exploration and in the broadly presented area of applications. The chaos in the role of irregular perturbations usually is considered as the negative aspect of systems dynamics and as the harmful process, and, therefore, it should be taking into account in the framework of the gyrostat-satellites attitude dynamics. Also the dynamical chaos can be used in its positive aspect as the dynamical instrument, which can change and improve dynamical processes (e.g. chaotic mixing of fuel components at the combustion in engines; the signals synchronization; the chaotic cryptography, etc.); and also dynamical chaos can be applied to the attitude control of satellites, that was considered in details in [34].

1. Main mechanical and mathematical models

Let us consider the angular motion (around the center of mass) of an axial gyrostat-satellite (dual-spin spacecraft) in the weak central gravity field at the initial coinciding of its angular momentum and the direction of the gravity force (fig.1).

The collinearity of the satellite angular momentum and the gravity field’s gradient can be realized in “conical precessions” regimes of a rotating prolate satellite, when it is stabilized by the gravitational way along the local vertical direction.

In this case, it is possible to approximately consider that the vector of the satellite angular momentum **K** is practically constant vector in the inertial space. This assumption allows to take the direction of the angular momentum vector as the short-term immovable direction coinciding with the action line of the gravity force applied to the center mass of the satellite (fig.1) – this direction of the local vertical is correspond to the axis *CZ* of the inertial coordinates frame *CXYZ*. The point *C* is the center of mass of the satellite.

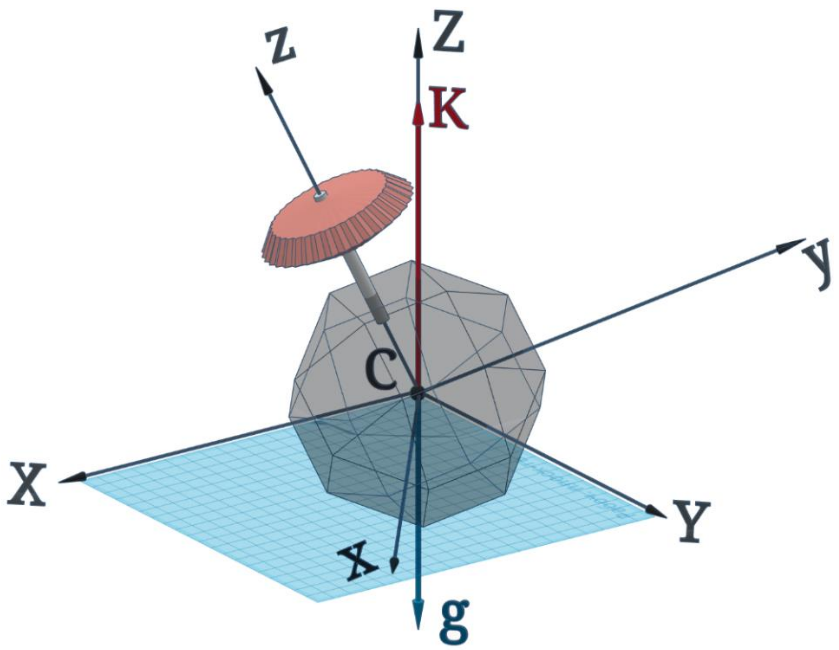


Fig.1 The schematic structure of the axial gyrostat with one rotor and the considered case of the spatial orientation of the system angular momentum (**K**) relatively the direction of the central gravity gradient vector (**g**)

The axial gyrostat is constructed as the system of two coaxial bodies: the main body with the general inertia tensor and the dynamically symmetrical rotor (such constructional scheme also is called as the “dual-spin spacecraft”). The main moving coordinates frame *Cxyz* is connected with the main body of the satellite; the axis *Cz* is the longitudinal axis of the satellite. The inertial axis *CZ*, as it was mentioned above, is anti-directional to the weak central gravity gradient vector **g** (i.e. the direction from the center of mass to the gravity center). The spatial position of the axis *CZ* can be described relatively the main body (fig.2) by the directional cosines $\gamma_1 = \cos(CZ, Cx)$, $\gamma_2 = \cos(CZ, Cy)$, $\gamma_3 = \cos(CZ, Cz)$. The rotor-body rotates relatively the main body, and it has its own absolute longitudinal angular momentum **Δ** (fig.2).

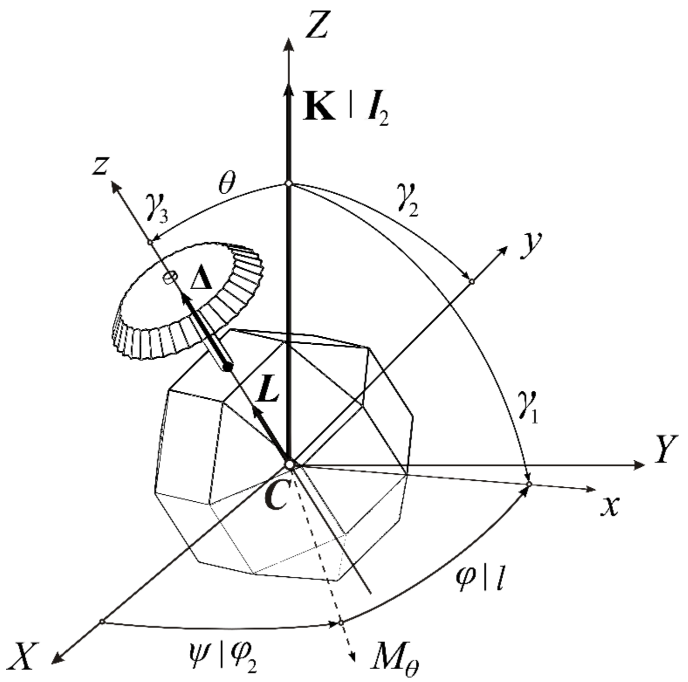


Fig.2 The axial gyrostat-satellite and coordinates frames

The dynamical equation of the angular motion can be written in the main moving coordinates frame *Cxyz* using the local derivation:

$$\frac{\tilde{d}}{dt}\mathbf{K} + \boldsymbol{\omega} \times \mathbf{K} = \mathbf{M}_g; \quad \dot{\Delta} = M_\Delta \tag{1.1}$$

where $\mathbf{M}_g$ – is the external torque from the central gravitational field, and $M_\Delta -$ is the internal torque acting on the rotor from the side of the main body along the axis *Cz* (e.g., it is a spin-up electromotor torque, a friction between bodies, etc.), and where the angular momentum of the system **K** in projections on axes of the connected coordinates frame *Cxyz* has the form:

$$\mathbf{K} = \left[Ap,\ Bq,\ C_b r + \Delta\right]^T \tag{1.2}$$

Here $A = A_b + A_r$, $B = B_b + A_r$, $C = C_b + C_r$; A_b, B_b, C_b are the axial inertia moments of the main body, and A_r, A_r, C_r are the axial inertia moments of the dynamically symmetrical rotor in the connected frame *Cxyz*; {*p*, *q*, *r*} – are the components of the angular velocity vector of the main body **ω** in the connected frame *Cxyz*. Also we assume that $A > B > C$ - this relation corresponds to the prolate form of the gyrostat (that is useful for the gravitational attitude stabilization).

The kinematical equations for the directional cosines can be written as the well-known Poisson's equations:

$$\begin{cases} \dot{\gamma}_1 = r\gamma_2 - q\gamma_3 \\ \dot{\gamma}_2 = p\gamma_3 - r\gamma_1 \\ \dot{\gamma}_3 = q\gamma_1 - p\gamma_2 \end{cases} \tag{1.3}$$

It is worth to indicate the well-known connections between the directional cosines and the Euler angles (fig.2):

$$\gamma_1 = \sin\theta \sin\varphi; \quad \gamma_2 = \sin\theta\cos\varphi; \quad \gamma_3 = \cos\theta \tag{1.4}$$

Let us investigate the axial gyrostat angular motion at the absence of the internal interaction between coaxial bodies (M_Δ=0), therefore, the value of the longitudinal angular momentum of the rotor is constant (Δ=const), that is corresponds to the natural attitude dynamics of the gyrostat-satellites (to the spin-stabilized motion of dual-spin spacecraft).

The external torque from the central gravitational field can be evaluated in the main moving frame *Cxyz* relatively the center of mass of the gyrostat (the point *C*) as the sum of its parts acting on the separate bodies. So, fulfilling the steps of general methodology [1], it is possible to write the following expressions for this torque (and its parts applied to the main body and to the rotor):

$$\mathbf{M}_g = \mathbf{M}_C^b + \mathbf{M}_C^r \tag{1.5}$$

Considering the dynamical symmetry of the rotor, the summands in (1.5) are:

$$\mathbf{M}_C^b = \mu \begin{bmatrix} (C_b - B_b)\gamma_2\gamma_3 \\ (A_b - C_b)\gamma_3\gamma_1 \\ (B_b - A_b)\gamma_1\gamma_2 \end{bmatrix}; \quad \mathbf{M}_C^r = \mu \begin{bmatrix} (C_r - A_r)\gamma_2\gamma_3 \\ (A_r - C_r)\gamma_3\gamma_1 \\ 0 \end{bmatrix} \tag{1.6}$$

where μ is the “gravitational parameter” depending on the distance between the gyrostat mass center and the center of gravity. Therefore, the common torque has the form:

$$\mathbf{M}_g = \mu \begin{bmatrix} (C-B)\gamma_2\gamma_3 \\ (A-C)\gamma_3\gamma_1 \\ (B-A)\gamma_1\gamma_2 \end{bmatrix}; \tag{1.7}$$

The dynamical equations of the axial gyrostat with one rotor in the central gravitational field (1.1) can be rewritten as follows:

$$\begin{cases} A\dot{p} + (C_b - B)qr + q\Delta = \mu(C-B)\gamma_2\gamma_3 \\ B\dot{q} + (A - C_b)pr - p\Delta = \mu(A-C)\gamma_3\gamma_1 \\ C_b\dot{r} + \dot{\Delta} + (B-A)pq = \mu(B-A)\gamma_1\gamma_2 \\ \dot{\Delta} = M_\Delta \end{cases} \tag{1.8}$$

Let us further investigate the gyrostat attitude dynamics in the weak central gravitation field at the initial “vertical” position of the angular momentum (fig.1, 2). In this case, the gravitational parameter μ has a small value and the corresponding external torque weakly affects the angular momentum of the system. At such assumptions, the angular momentum (with its quite large value) practically does not change its vertical orientation (**K**≈const) under the influence of the perturbing gravitational torque. Then it is possible to define approximately the directional cosines of the "vertical" axis *CZ* using the components of the angular momentum:

$$\gamma_1 = Ap/K; \quad \gamma_2 = Bq/K; \quad \gamma_3 = (C_b r + \Delta)/K \tag{1.9}$$

The substitution of expressions (1.9) into equations (1.8) allows to write the closed form of differential equations at the absence of the interaction between the main body and the rotor of the gyrostat (M_Δ=0):

$$\begin{cases} A\dot{p} + (C_b - B)qr + q\Delta = \dfrac{\mu}{K^2}(C - B)Bq(C_b r + \Delta) \\ B\dot{q} + (A - C_b)pr - p\Delta = \dfrac{\mu}{K^2}(A - C)Ap(C_b r + \Delta) \\ C_b\dot{r} + (B - A)pq = \dfrac{\mu}{K^2}(B - A)ABpq \\ \Delta = \text{const} \end{cases} \tag{1.10}$$

As it follows from the right parts of the equations (1.10), the action of the central gravity torque can be formally considered as the small perturbation, which do not sufficiently change the angular momentum of the torque-free system when the following dimensionless parameter is small:

$$\varepsilon = \frac{\mu AB}{K^2} << 1 \tag{1.11}$$

The condition (1.11) supplies the fulfillment of the approximations (1.9) and the correctness of the considering task of the gyrostat-satellite motion in the weak central gravitational field.

2. The connection of the considered gyrostat motion with the V.A. Stekloff case

As it is known, V.A. Stekloff showed [5] that the solution of F. Brune [6] is the partial case of the solution of the equations of a body motion in a liquid, which can be presented in the reduced and generalized form, that is appropriate for gyrostat model [7] (in the designations from [7]):

$$\begin{cases} \left\{ A_1 \dfrac{d\omega_1}{dt} = (A_2 - A_3)(\omega_2\omega_3 - \varepsilon R_2 R_3) + \lambda_2\omega_3 - \lambda_3\omega_2 + \mu_2 R_3 - \mu_3 R_2 \right\}_{(1,2,3)} \\ \left\{ \dfrac{dR_1}{dt} = \omega_3 R_2 - \omega_2 R_3 \right\}_{(1,2,3)} \end{cases} \tag{2.1}$$

where the subscription (1,2,3) means the circular permutation in all low indexes; parameters λ_i and μ_i are defined by cyclical flows of the liquid (they equal to zero at all zero main circulations); variables R_i represent complexes of "impulsive forces" and circulations parameters.

If we express the absolute angular momentum of the rotor through its relative angular velocity, then the system (1.8) takes the form:

$$\begin{cases} A\dot{p} + (C-B)qr + qC_r\sigma = \mu(C-B)\gamma_2\gamma_3 \\ B\dot{q} + (A-C)pr - pC_r\sigma = \mu(A-C)\gamma_3\gamma_1 \\ C\dot{r} + C_r\dot{\sigma} + (B-A)pq = \mu(B-A)\gamma_1\gamma_2 \\ C_r(\dot{r} + \dot{\sigma}) = M_\Delta \end{cases} \tag{2.2}$$

where σ is the relative angular velocity of the rotor and M_Δ is the internal torque acting on the rotor from the side of the main body.

Passing to our notations, and taking that $\{\lambda_1 = \lambda_2 = 0;\ \lambda_3 = C_r\sigma;\ \mu_i = 0;\ \varepsilon = \mu;\ R_i = \gamma_i\}$ the system (2.1) is strictly reduces to (2.2) and (1.3) at the condition that the relative angular momentum of the rotor is constant $(\lambda_3 = C_r\sigma = \text{const})$, that can be fulfilled only at the action of the followed "stabilizing" internal torque:

$$M_\Delta = C_r\dot{r} \tag{2.3}$$

Therefore, from the system (2.1) the equations of the gyrostat motion with the constant relative angular momentum of the rotor (at the internal torque (2.3)) in the central gravity field follow, i.e. the task of the motion of the body/gyrostat in liquid and task of the motion of the body/gyrostat in the central gravity field are in a sense analogous – it is, in fact, the formulation of the *analogy* of V.A. Stekloff with its expansion by P.V. Kharlamov on the gyrostat case:

$$\begin{cases} \begin{cases} A\dot{p} + (C-B)qr + q\lambda_3 = \mu(C-B)\gamma_2\gamma_3 \\ B\dot{q} + (A-C)pr - p\lambda_3 = \mu(A-C)\gamma_3\gamma_1 \\ C\dot{r} + (B-A)pq = \mu(B-A)\gamma_1\gamma_2 \end{cases} \\ \begin{cases} \dot{\gamma}_1 = r\gamma_2 - q\gamma_3 \\ \dot{\gamma}_2 = p\gamma_3 - r\gamma_1 \\ \dot{\gamma}_3 = q\gamma_1 - p\gamma_2 \end{cases} \end{cases} \tag{2.4}$$

As it is known [1], in the Stekloff case the system equations for the rigid body motion has the partial solution in the form

$$\gamma_1 = A'p; \quad \gamma_2 = B'q; \quad \gamma_3 = C'r \tag{2.5}$$

were A', B', C' are some constant; and then the dynamical system can be reduced to the torques-free system with the corresponded solution for the free rigid body.

In the case of the gyrostat motion with the constant relative angular momentum of rotors (λ_i=const) in central gravity field, in the accordance to the generalized analogy of V.A. Stekloff – P.V. Kharlamov, the analytical solution was obtained by P.V. Kharlamov [8].

Returning to our case of the gyrostat motion at the absence of the internal interaction and at the constancy of the absolute longitudinal angular momentum of the rotor $(\Delta = \text{const};\ M_\Delta = 0)$, we should note that the P.V. Kharlamov's solution [8] is not appropriate, and another solution should be obtained for this case. In this connection, let us obtain such solution as the analytical solution of the equations (1.10).

Thus, in view of the above-mentioned aspects of the V.A. Stekloff – P.V. Kharlamov analogy, and due to the symmetry of the assumptions (2.5) and (1.9), the following in the next section analytical solutions of the equations (1.10) can be considered as a partial generalization of the V.A. Stekloff case on the gyrostat motion.

3. The analytical solution for dynamical parameters

In purposes of the analytical investigation of the axial gyrostat motion dynamics it is needed to write the so-called “first” integrals. If we multiply the first equation (1.10) by *Ap*, the second – by *Bq*, third – by ($C_b r+\Delta$), then after the summation of results, the expression follows:

$$\frac{d}{dt}\left[A^2p^2+B^2q^2+\left(C_b r+\Delta\right)^2\right]=0$$

that can be rewritten after integrating as the “first” integral for the angular momentum value:

$$A^2p^2+B^2q^2+\left(C_b r+\Delta\right)^2=\mathrm{const}=K^2 \tag{3.1}$$

The energy conservation law can be presented as the sum of kinetic and potential energy:

$$E=T+P=\frac{1}{2}\left(Ap^2+Bq^2+C_b r^2+\frac{\Delta^2}{C_r}\right)+\frac{\mu}{2}\left(A\gamma_1^2+B\gamma_2^2+C\gamma_3^2\right)=h=\mathrm{const} \tag{3.2}$$

Using (1.9) from the last expression the “first” integral follows:

$$Ap^2\left(1+\frac{\mu A^2}{K^2}\right)+Bq^2\left(1+\frac{\mu B^2}{K^2}\right)+C_b r^2+\frac{\Delta^2}{C_r}+\frac{\mu C\left(C_b r+\Delta\right)^2}{K^2}=2h \tag{3.3}$$

Multiplying (3.3) by $A\big/\left[1+\mu A^2/K^2\right]$ and deducting (3.1), we obtain:

$$\left(AB\frac{K^2+\mu B^2}{K^2+\mu A^2}-B^2\right)q^2+\frac{AK^2}{K^2+\mu A^2}\left[C_b r^2+\frac{\Delta^2}{C_r}+\frac{\mu C\left(C_b r+\Delta\right)^2}{K^2}\right]-\\-\left(C_b r+\Delta\right)^2=\frac{2hAK^2}{K^2+\mu A^2}-K^2 \tag{3.4}$$

Analogously, multiplying (3.3) by $B\big/\left[1+\mu B^2/K^2\right]$ and deducting (3.1), we write:

$$\left(AB\frac{K^2+\mu A^2}{K^2+\mu B^2}-A^2\right)p^2+\frac{BK^2}{K^2+\mu B^2}\left[C_b r^2+\frac{\Delta^2}{C_r}+\frac{\mu C\left(C_b r+\Delta\right)^2}{K^2}\right]-\\-\left(C_b r+\Delta\right)^2=\frac{2hBK^2}{K^2+\mu B^2}-K^2 \tag{3.5}$$

The groupment of complete squares in expressions (3.4) and (3.5) allows to rewrite:

$$B\left(A\frac{K^2+\mu B^2}{K^2+\mu A^2}-B\right)q^2+a_\alpha\left\{D-\frac{\alpha\Delta}{a_\alpha C_b}\right\}^2=H \tag{3.6}$$

$$-A\left(A-B\frac{K^2+\mu A^2}{K^2+\mu B^2}\right)p^2+b_\beta\left\{D-\frac{\beta\Delta}{b_\beta C_b}\right\}^2=G \tag{3.7}$$

where the local designation is involved

$$D = C_b r + \Delta$$

and the following coefficients are used:

$$\begin{cases} \alpha = \dfrac{AK^2}{K^2+\mu A^2}; \quad a_\alpha = \dfrac{\alpha}{C_b} + \mu \dfrac{\alpha C}{K^2} - 1; \quad \beta = \dfrac{BK^2}{K^2+\mu B^2}; \quad b_\beta = \dfrac{\beta}{C_b} + \mu \dfrac{\beta C}{K^2} - 1; \\ c_\alpha = \Delta^2 \left(\dfrac{\alpha^2}{a_\alpha C_b^2} - \dfrac{\alpha}{C_b} - \dfrac{\alpha}{C_r} \right); \quad H = c_\alpha + \dfrac{2hAK^2}{K^2+\mu A^2} - K^2; \\ d_\beta = \Delta^2 \left(\dfrac{\beta^2}{b_\beta C_b^2} - \dfrac{\beta}{C_b} - \dfrac{\beta}{C_r} \right); \quad G = d_\beta + \dfrac{2hBK^2}{K^2+\mu B^2} - K^2 \end{cases} \tag{3.8}$$

From (3.6) the expression follows:

$$D - \frac{\alpha\Delta}{a_\alpha C_b} = \pm V(q); \quad V(q) = \sqrt{\frac{1}{a_\alpha}\left(H - B\left(A\frac{K^2+\mu B^2}{K^2+\mu A^2} - B \right) q^2 \right)} \tag{3.9}$$

The value in curly brackets in (3.7) can be rewritten

$$D - \frac{\beta\Delta}{b_\beta C_b} = \left\{ D - \frac{\alpha\Delta}{a_\alpha C_b} \right\} - \frac{\Delta}{C_b}\left[\frac{\beta}{b_\beta} - \frac{\alpha}{a_\alpha} \right]$$

and, therefore, the expression is correct:

$$D - \frac{\beta\Delta}{b_\beta C_b} = \pm V(q) - \frac{\Delta}{C_b}\left[\frac{\beta}{b_\beta} - \frac{\alpha}{a_\alpha} \right] \tag{3.10}$$

Then from (3.7) it is possible to write:

$$p = \pm W(q); \quad W(q) = \sqrt{\frac{b_\beta \left\{ \pm V(q) - \dfrac{\Delta}{C_b}\left[\dfrac{\beta}{b_\beta} - \dfrac{\alpha}{a_\alpha} \right] \right\}^2 - G}{A\left(A - B\dfrac{K^2+\mu A^2}{K^2+\mu B^2} \right)}} \tag{3.11}$$

The second equation (1.10) can be transformed to the form:

$$B\dot{q} + p\left\{ \frac{A}{C_b} - 1 - \frac{\mu}{K^2}(A-C)A \right\}\left(D - \frac{\alpha\Delta}{a_\alpha C_b} \right) = 0$$

that after using (3.11) and (3.9) takes the shape:

$$B\dot{q} = \pm f \cdot W(q)V(q); \quad f = \left\{ \frac{A}{C_b} - 1 - \frac{\mu}{K^2}(A-C)A \right\} \tag{3.12}$$

Now to integrating (3.12) the change of variables can be used:

$$x = \pm V(q) - \frac{\Delta}{C_b}\left[\frac{\beta}{b_\beta} - \frac{\alpha}{a_\alpha}\right] \tag{3.13}$$

After variables changing (3.13), for the equation (3.12) we obtain the canonical form for the following elliptic integrals:

$$dt = \pm \frac{M}{\sqrt{ac}} \frac{dx}{\sqrt{\left(\sqrt{\frac{H}{a}}\right)^2 - (x+b)^2}\sqrt{x^2 - \left(\sqrt{\frac{G}{c}}\right)^2}} \tag{3.14}$$

where constants and initial values are:

$$M = \frac{a_\alpha B}{f}\sqrt{\frac{A\left(A - B\frac{K^2+\mu A^2}{K^2+\mu B^2}\right)}{B\left(A\frac{K^2+\mu B^2}{K^2+\mu A^2} - B\right)}}; \quad a = a_\alpha; \quad b = \frac{\Delta}{C_b}\left[\frac{\beta}{b_\beta} - \frac{\alpha}{a_\alpha}\right]; \quad c = b_\beta \tag{3.15}$$

$$x(t_0) = x_{ini} = \pm V(q_0) - \frac{\Delta}{C_b}\left[\frac{\beta}{b_\beta} - \frac{\alpha}{a_\alpha}\right] \tag{3.16}$$

The second variables change allows to write the equation (3.14) in the form

$$dt = \pm 2eM\frac{\sqrt{R/P}}{\sqrt{aG}}\left[\sqrt{s_2 s_4}\sqrt{\left(1 - \frac{z^2}{c_1^2}\right)\left(1 - \frac{z^2}{c_2^2}\right)}\right]^{-1} dz; \qquad z = \sqrt{\frac{R(x-e)}{P(x+e)}}; \tag{3.17}$$

where

$$R = -b - d + e; \; P = -b - d - e; \; d = \sqrt{H/a}; \; e = \sqrt{G/c}; \; c_1^2 = s_2/s_1; \; c_2^2 = s_4/s_3;$$

$$s_1 = d + e - b; \; s_2 = \frac{R}{P}[d - e - b]; \; s_3 = d - e + b; \; s_4 = \frac{R}{P}[d + e + b];$$

After one more changing variables $\left(z = \tilde{c}y; \;\; \tilde{c} = \min\{c_1, c_2\}; \;\; \underset{\sim}{c} = \max\{c_1, c_2\}; \;\; k = \tilde{c}/\underset{\sim}{c}\right)$ the equation (3.17) can be integrated as the elliptic integral:

$$\pm\left[N(t - t_0) + I_0\right] = \int_0^y \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}; \tag{3.18}$$

where

$$N = \left[2eM\frac{\tilde{c}\sqrt{R/P}}{\sqrt{aG}\sqrt{s_2 s_4}}\right]^{-1}; \;\; I_0 = \int_0^{y_0} \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}} = \text{const}$$

Then the inversion of the elliptic integral gives the analytical solution in the form of the Jacobi elliptic sine

$$y(t)=\operatorname{sn}\left[\pm\left(N(t-t_0)+I_0\right),k\right] \tag{3.19}$$

Fulfilling the back variables change, we obtain analytical solutions for all dynamical parameters:

$$\begin{cases} p(t)=\pm\sqrt{\dfrac{cx^2-G}{A\left(A-B\dfrac{K^2+\mu A^2}{K^2+\mu B^2}\right)}}; \quad q(t)=\pm\sqrt{\dfrac{H-a\left(x(t)+b\right)^2}{B\left(A\dfrac{K^2+\mu B^2}{K^2+\mu A^2}-B\right)}}; \\ r(t)=\dfrac{1}{C_b}\left[x(t)+\dfrac{\beta\Delta}{b_\beta C_b}-\Delta\right]; \quad x(t)=e\dfrac{R/P+\tilde{c}^2\operatorname{sn}^2\left[\pm\left(N(t-t_0)+I_0\right),k\right]}{R/P-\tilde{c}^2\operatorname{sn}^2\left[\pm\left(N(t-t_0)+I_0\right),k\right]} \end{cases} \tag{3.20}$$

So, finally we have the exact explicit analytical solution (3.20) for the parameters of the gyrostat angular motion around the center of mass in the central gravity field at the initial "vertical" position of the angular momentum. This solution fully describes the attitude dynamics of the gyrostat in cases of conical precessions at the implementation of the gravitational stabilized motion (at least on a quite short time-interval of the orbital motion, when the rotation of the local vertical is inessential).

To check the correctness of the solution it is possible to present the comparative results of the analytical and numerical modelling. At the figure (fig.3) is shown the time-history of the angular velocity components and directional cosines calculated by numerical integrating and with the help of the analytical solution (3.20). The numerical integration was fulfilled using the full equations (1.8), (1.3). The systems parameters and initial conditions are presented in the table (tabl.1) – these parameters correspond to the motion of micro-gyrostat-satellites (with masses from 10 to 100 kg) in the central gravity field of the Earth[1]; for all calculations in this paper the following inertia moments were taken: A_b=15, B_b=10, C_b=7, A_r=5, C_r=4 [kg·m^2].

As can we see from the figure (fig.3-a, b), there is the complete coincidence of numerical results of the full equations (1.8), (1.3) and analytical solutions (3.20), (1.9) at the large natural μ-values. Certainly, this coincidence is gradually violated with the growth of the parameter μ up to super large hypothetical values (fig.3-c, d).

4. The canonical form of the system in the Serret-Andoyer variables

For better understanding dynamical properties of the system, it is possible to use the research tools of the Hamiltonian formalism together with the canonical variables of Serret-Andoyer [1]. As it is known, the angular momentum components are linked with the canonical Serret-Andoyer momentums as follows (fig.2):

$$K_x=Ap=\sqrt{I_2^2-L^2}\sin l; \quad K_y=Bq=\sqrt{I_2^2-L^2}\cos l; \quad K_z=C_b r+\Delta=L; \quad K=I_2 \tag{4.1}$$

In the considering case, using the expressions (1.9), we can write the following expressions for the directional cosines through the Serret-Andoyer variables:

$$\gamma_1=\frac{\sqrt{I_2^2-L^2}}{I_2}\sin l; \quad \gamma_2=\frac{\sqrt{I_2^2-L^2}}{I_2}\cos l; \quad \gamma_3=\frac{L}{I_2} \tag{4.2}$$

---

[1] The values μ=(0.1÷0.2)·10^{-7} [1/s^2] correspond to geostationary/high Earth-orbits; and the values (0.4÷0.5)·10^{-5} [1/s^2] – to the low Earth-orbits/surface.

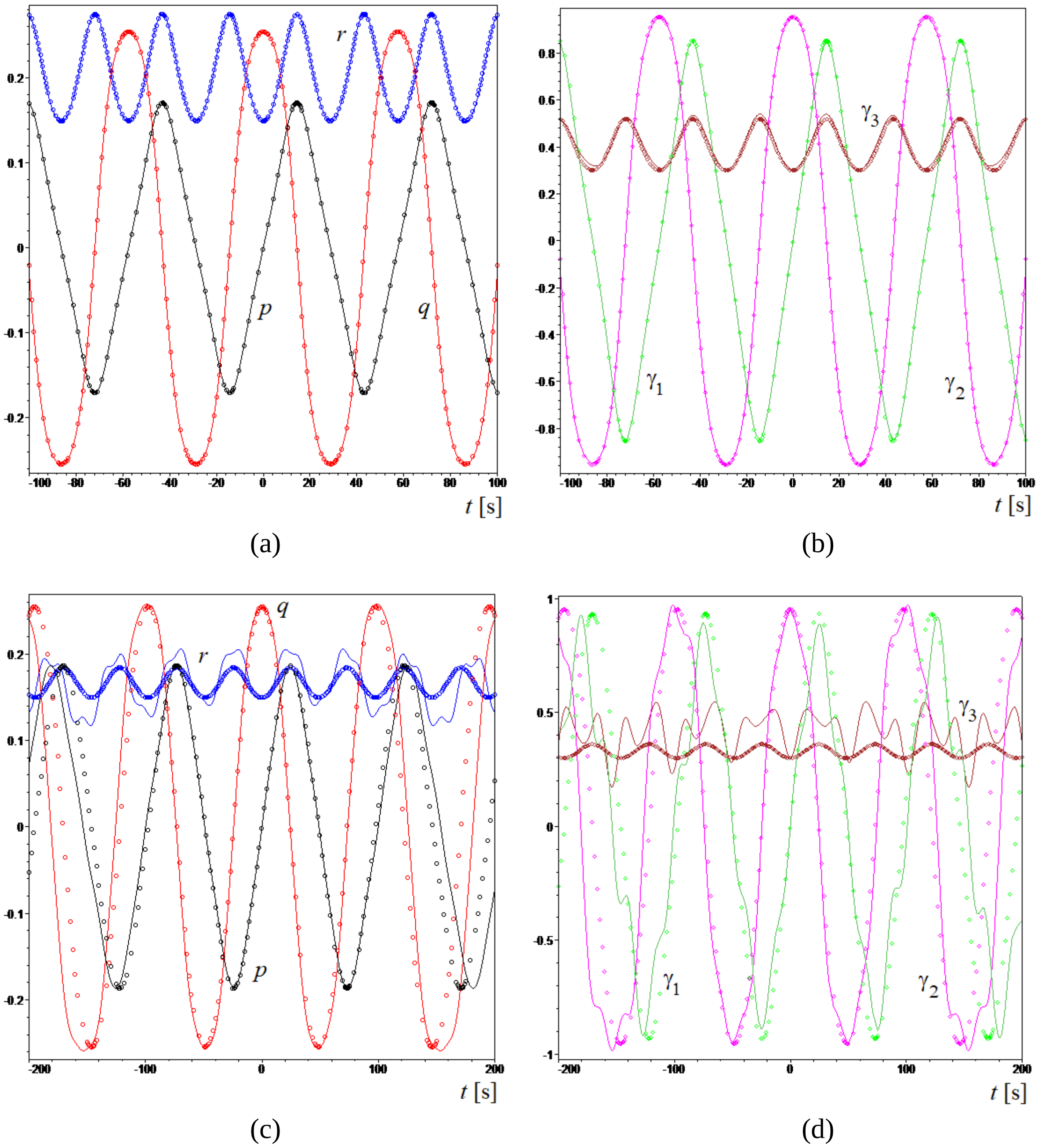


Fig.3. The time-history of dynamical parameters:
lines – numerical modelling for the full system ((1.8), (1.3)),
dots – analytical solutions (3.20) for the reduced approximate system ((1.9), (1.10))

Then the Hamiltonian of the system in the Serret-Andoyer variables has the form, which follows from the general form (3.2) of the full energy:

$$\mathcal{H} = T + P;$$

$$T = \frac{I_2^2 - L^2}{2}\left[\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right] + \frac{1}{2}\left[\frac{\Delta^2}{C_r} + \frac{(L-\Delta)^2}{C_b}\right]; \tag{4.3}$$

$$P = \frac{\mu}{2I_2^2}\left\{\left(I_2^2 - L^2\right)\left[A\sin^2 l + B\cos^2 l\right] + CL^2\right\}$$

where T is the kinetic energy; P is the potential energy. The Hamiltonian (4.3) corresponds to the full energy of the conservative system, and, therefore $\mathcal{H} = E = \text{const}$.

The system in the considering case has only one positional degree of freedom (other coordinates are cyclic):

$$\begin{cases} \dot{L} = \dfrac{1}{2AB}\left(1 - \dfrac{\mu AB}{I_2^2}\right)\left(I_2^2 - L^2\right)\left(A - B\right)\sin 2l \\ \dot{l} = L\left[\dfrac{1}{C_b} + \dfrac{\mu C}{I_2^2} - \left(\dfrac{1}{A} + \dfrac{\mu A}{I_2^2}\right)\sin^2 l - \left(\dfrac{1}{B} + \dfrac{\mu B}{I_2^2}\right)\cos^2 l\right] - \dfrac{\Delta}{C_b} \end{cases} \tag{4.4}$$

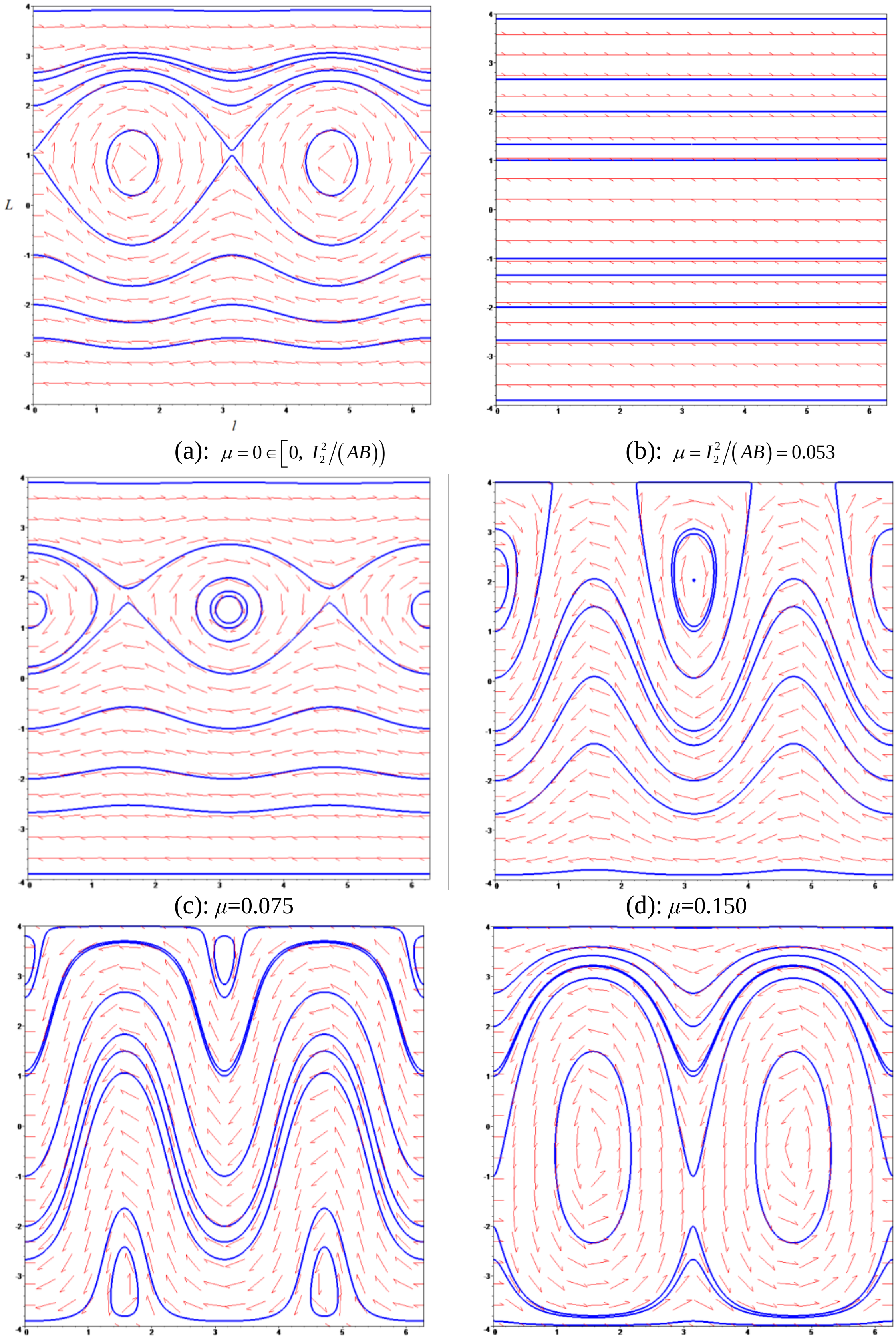
The qualitative forms of the phase space of the system (4.4) can be presented (fig.4) for different μ-values, starting from the zero and up to super large hypothetical magnitudes, which already do not correspond to the natural values of the gravitational parameters of the Earth. This set of different forms of phase portraits (fig.4) shows the series of bifurcations at the gradual growth of the μ-value, and it is interesting from the theoretical point of view in the framework of the complete qualitative analysis. In addition, we ought to note, that for natural values of the gravitational parameters $(0 < \mu << 1)$ only the first type of the phase portrait (fig.4-a) is appropriate; and other types of the phase portrait are hypothetical (corresponded to attractive centers with super-large gravity).

The Hamiltonian conservative system (4.4) correctly and simply describe the basic dynamical behavior of the gyrostat in the central gravity field at the assumptions of the smallness of the gravitational torque and at the coincidence of the gravity vector with the angular momentum direction (that is finally expressed in the relations (1.9)). This behavior is regular (fig.4); it can be taken as the main unperturbed dynamics in the framework of the rough consideration of the gyrostat-satellite motion in the framework of a primary design of space missions.

However, with an increase in deviations from the above-mentioned assumptions, we should return to using the general equations (1.8) and (1.3) to describe the more complicated dynamics, which can demonstrate some hidden subtle aspect, including the irregular behavior. The example of such irregular behavior is presented at the figure (fig.5). There we see chaotic time-histories of all dynamical parameters (fig.5-a, b), the chaotic time-dependency of the small change (not exceeding 5÷10%) of the relative value of the angular momentum (fig.5-c), and the complex phase-trajectory with the entanglement of its shape (fig.5-d).

5. The chaotization analysis

To understand above-mentioned irregular phenomena we should write the equations for the Serret-Andoyer variables without the simplification (1.9) at the start of modeling. In this case the system still remains the Hamiltonian type since in the potential central gravity field the full energy is constant, and the Hamiltonian of the system can be written based on expression (3.2) after the introduction of the canonical variables. As it is clear, the system has three degree of freedom, therefore three independent canonical coordinates and three corresponding independent canonical impulses can be introduced with the help of the well-known procedure and conjunctional expressions. However, let us deviate from this classical approach, and write the closed system of equations for a part of canonical coordinates/impulses and directional cosines.



(a): $\mu = 0 \in \left[0,\ I_2^2/(AB)\right)$ (b): $\mu = I_2^2/(AB) = 0.053$

(c): μ=0.075 (d): μ=0.150

(e): μ=0.210 (f): μ=0.500

Fig.4. The set of phase portraits $\{l, L\}$ at the growth of the gravitational parameter μ [1/s^2]

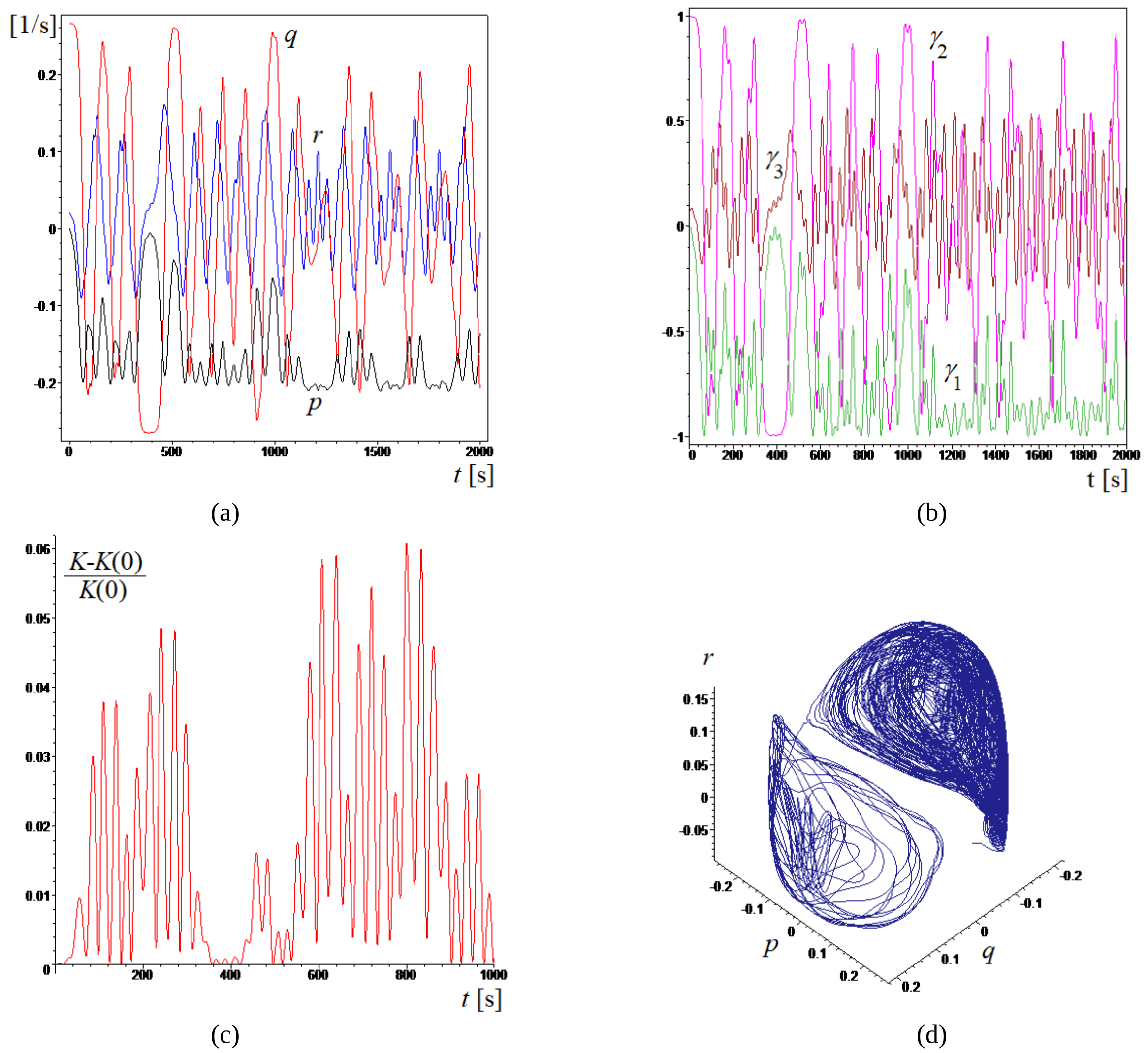


Fig.5. The chaotic regime of the system

The direct differentiation of the expressions (4.1) and some simplifications allow to obtain the following formulae:

$$\begin{cases} \dot{l} = \dfrac{1}{Bq}\left\{ A\dot{p} - \dfrac{Ap}{A^2p^2 + B^2q^2}\left[A^2p\dot{p} + B^2q\dot{q} \right] \right\}; \\ \dot{L} = C_b\dot{r}; \\ \dot{I}_2 = \dfrac{1}{I_2}\left\{ \left[A^2p\dot{p} + B^2q\dot{q} \right] + \left(C_br + \Delta \right) C_b\dot{r} \right\} \end{cases} \tag{5.1}$$

Substituting the expression for derivations of the angular velocities from the main dynamical equations (1.8) into (5.1) and using again (4.1), we can write

$$\begin{cases} \dot{l} = L\left[\dfrac{1}{C_b} - \dfrac{\sin^2 l}{A} - \dfrac{\cos^2 l}{B} \right] - \dfrac{\Delta}{C_b} + \mu\dfrac{\gamma_3}{\sqrt{I_2^2 - L^2}}\left[(C-B)\gamma_2\cos l - (A-C)\gamma_1\sin l \right]; \\ \dot{L} = \dfrac{A-B}{2AB}\left(I_2^2 - L^2 \right)\sin 2l + \mu(B-A)\gamma_1\gamma_2; \\ \dot{I}_2 = \dfrac{\mu}{I_2}\left\{ \sqrt{I_2^2 - L^2}\left[(C-B)\gamma_2\sin l + (A-C)\gamma_1\cos l \right]\gamma_3 + L(B-A)\gamma_1\gamma_2 \right\}; \end{cases} \tag{5.2}$$

In addition, passing from the angular velocity components to the Serret-Andoyer variables we rewrite the kinematical equations (1.3):

$$\begin{cases} \dot{\gamma}_1 = \dfrac{L-\Delta}{C_b}\gamma_2 - \dfrac{\sqrt{I_2^2 - L^2}\cos l}{B}\gamma_3; \\ \dot{\gamma}_2 = \dfrac{\sqrt{I_2^2 - L^2}\sin l}{A}\gamma_3 - \dfrac{L-\Delta}{C_b}\gamma_1; \\ \dot{\gamma}_3 = \sqrt{I_2^2 - L^2}\left[\dfrac{\cos l}{B}\gamma_1 - \dfrac{\sin l}{A}\gamma_2\right] \end{cases} \tag{5.3}$$

The equations (5.2) and (5.3) together form the closed complete dynamical system fully describing the dynamics of the gyrostat in all details and aspects via the set of mixed variables $\{l, L, I_2, \gamma_1, \gamma_2, \gamma_3\} \in \mathbb{R}^6$. It is worth to note, that at the substitution of expressions-assumptions (1.9) into the system (5.2), we obtain the equations (4.4), and herewith the third equation (5.2) is reduced to the shape $I_2 \equiv 0$.

As it is possible to see from the structure of the equations system (5.2), the action of the gravitational torques can be considered as the perturbation of the torques-free (μ=0) unperturbed dynamics of the free gyrostat. This unperturbed dynamics is fully described by the canonical coordinates $\{l, L\}$ at the predefined constant value of the angular momentum $I_2 = K$ and, therefore, the phase space of the unperturbed system can be represented as the phase plane $\{0 \le l \le 2\pi, -1 \le L/I_2 \le 1\}$, and this is depicted at the figure (fig.6-a), where colors of the trajectories symbolize different levels of the energy (the black color corresponds to the heteroclinic separatrix-trajectories).

As it is known (e.g. [29]) the torques-free system (μ=0) has the heteroclinic solution damping to the constant values of the dynamical parameters, corresponding to stationary saddle-points. These saddle-points have the coordinates [l={0, π, 2π}; L=L_s] in the Serret-Andoyer phase space, and, that is the same, the coordinates [p=0; q=q_s, r=r_s] in the phase space of the angular velocity components (with constant values q_s and r_s). At the substitution these constants into the system (1.3), we obtain the linear differential equations describing partial regimes of motion close to the stationary saddle-points:

$$\begin{cases} \dot{\bar{\gamma}}_1 = r_s\bar{\gamma}_2 - q_s\bar{\gamma}_3 \\ \dot{\bar{\gamma}}_2 = -r_s\bar{\gamma}_1 \\ \dot{\bar{\gamma}}_3 = q_s\bar{\gamma}_1 \end{cases} \tag{5.4}$$

From the system (5.4) the linear equation follows:

$$\ddot{\bar{\gamma}}_1 = -\left(r_s^2 + q_s^2\right)\bar{\gamma}_1 \tag{5.5}$$

The solution of the equation (5.5) has the simplest harmonic structure:

$$\bar{\gamma}_1(t) = D_1 \sin\Omega t + D_2 \cos\Omega t; \quad \Omega = \sqrt{r_s^2 + q_s^2} \tag{5.6}$$

After integrating the second and the third equations (5.4) it is possible to conclude that all of the partial solutions $\{\bar{\gamma}_1(t), \bar{\gamma}_2(t), \bar{\gamma}_3(t)\}$ corresponding to heteroclinic saddle-points have the exact harmonic structure. So, the harmonic structure of the directional cosines solutions will remain for all possible saddle-points, limiting the all heteroclinic trajectories in the phase space. This circumstance allows to approximately consider the time-dependencies for the directional cosines along heteroclinic trajectories as more compound functions of time, which, nevertheless, are decomposable and representable in the form of polyharmonic functions:

$$\bar{\boldsymbol{\gamma}}(t) = \boldsymbol{\eta}(t) \tag{5.7}$$

where $\boldsymbol{\eta}(t) = [\eta_1(t), \eta_2(t), \eta_3(t)]^T$ is the vector of polyharmonic functions of time, and $\bar{\boldsymbol{\gamma}} = \left[\bar{\gamma}_1(t)\big|_{\overline{S_i S_j}},\ \bar{\gamma}_2(t)\big|_{\overline{S_i S_j}},\ \bar{\gamma}_3(t)\big|_{\overline{S_i S_j}} \right]^T$; the subscription $\overline{S_i S_j}$ symbolizes the correspondence of the solution to initial conditions of the realization of the unperturbed heteroclinic trajectory, which is limiting by two saddles S_i and S_j (fig.6-a). Here we note that exact explicit shapes of the functions $\boldsymbol{\eta}(t)$ are not important and necessary for the further analysis; it is enough to know that these functions have polyharmonic structure.

Now to describe the perturbed dynamics near the heteroclinic trajectories we can formally rewrite the equations (5.2) in the classical form of the perturbed system with small polyharmonic perturbations [e.g. 29]:

$$\begin{cases} \dot{l} = f_l(l, L) + \varepsilon g_l(l, L, t); \\ \dot{L} = f_L(l, L) + \varepsilon g_L(l, L, t); \\ \dot{I}_2 = \varepsilon g_{I_2}(l, L, t); \end{cases} \tag{5.8}$$

with the following right-parts functions:

$$\begin{cases} f_l(l, L) = L\left[\dfrac{1}{C_b} - \dfrac{\sin^2 l}{A} - \dfrac{\cos^2 l}{B} \right] - \dfrac{\Delta}{C_b}; \\ f_L(l, L) = \dfrac{A - B}{2AB}\left(\bar{I}_2^2 - L^2 \right)\sin 2l; \\ g_l = \dfrac{\bar{I}_2^2}{AB} \dfrac{\eta_3(t)}{\sqrt{\bar{I}_2^2 - L^2}} \left[(C - B)\eta_2(t)\cos l - (A - C)\eta_1(t)\sin l \right] + \mathrm{O}\left(\varepsilon^2\right); \\ g_L(l, L, t) = \dfrac{\bar{I}_2^2}{AB}(B - A)\eta_1(t)\eta_2(t) + \mathrm{O}\left(\varepsilon^2\right); \\ g_{I_2}(l, L, t) = \dfrac{\bar{I}_2}{AB}\left\{ \sqrt{\bar{I}_2^2 - L^2}\left[(C - B)\eta_2(t)\sin l + (A - C)\eta_1(t)\cos l \right]\eta_3(t) + \right. \\ \left. + L(B - A)\eta_1(t)\eta_2(t) \right\} + \mathrm{O}\left(\varepsilon^2\right); \end{cases} \tag{5.9}$$

where $\bar{I}_2 = K$ and $\mathrm{O}\left(\varepsilon^2\right)$ are terms proportional to the second (and higher) powers of the small parameter.

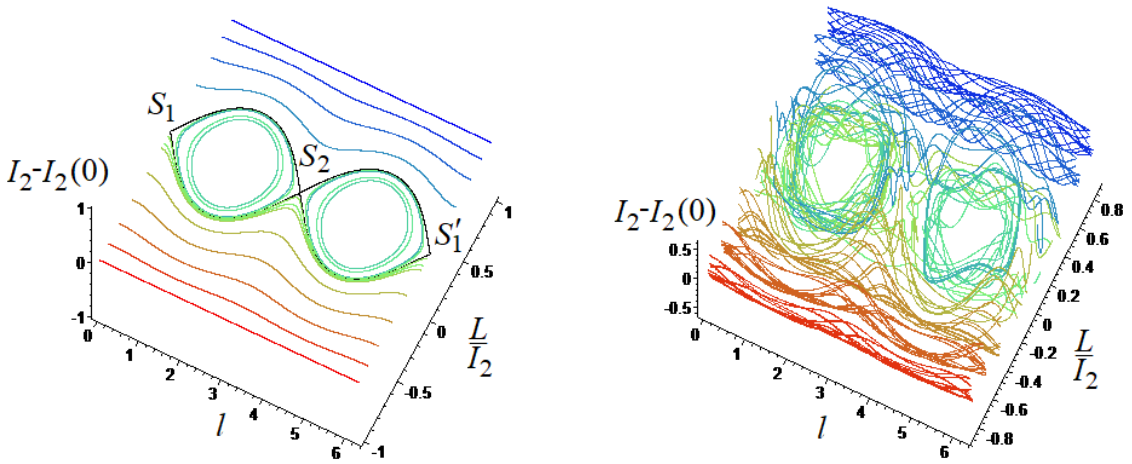
Then the subsystem of the first and the second equations (5.8) can be considered separately from the third equation; this subsystem is the single degree-of-freedom system perturbed system with polyharmonic perturbations. The similar perturbed subsystems with the polyharmnic perturbations have been investigated in many articles in the framework of the gyrostat chaotic dynamics analysis [e.g. 29].

Not repeating similar studies, we can say at once that the perturbed system (5.8) will demonstrate the chaotic dynamics near the heteroclnic trajectories of the unperturbed system of the torque-free gyrostat due to the heteroclinic nets initiations in the gyrostat phase space under the action of small polyharmonic perturbations (and without regularizing factors like frictions, the energy dissipation/excitation, etc.). This statement can be confirmed by the way of the Melnikov function evaluating, which in the considered case will have the polyhamonic form by analogy with [29], that defines the infinite number of intersections of splitted stable/unstable manifolds of heteroclinic trajectories and the heteroclinic net and chaos generation:

$$M(t_0) = \varepsilon \int_{-\infty}^{+\infty} \left[f_L g_l - f_l g_L \right]_{(\bar{l}(t), \bar{L}(t), \boldsymbol{\eta}(t+t_0))} dt = \text{polyharm}(t_0) \tag{5.10}$$

Also the chaotic behavior can be demonstrated by the way of the Poincaré sections plotting – we will present the Poincaré sections for the full dynamical system {(5.2) and (5.3)} without its localizations near the heteroclinic area.

So, the detected and confirmed chaotic dynamics of the gyrostat in the central gravitation field (based on the consideration of the localized system (5.8)) will look sufficiently more complicated at the modelling with the help of the full dynamical equations {(5.2) and (5.3)}; and the especial complication occurs in the area of heteroclinic trajectories (fig.6-b). To show this complex chaotic dynamics it is appropriate to plot the full-dimensional phase portraits (fig.7) of the system (5.2) and their Poincaré sections by the plane I_2=$I_2(0)$=K (fig.8).



(a): μ=0 (b): μ=0.046

Fig.6. The change of the phase portrait structure under the action of the gravity torques

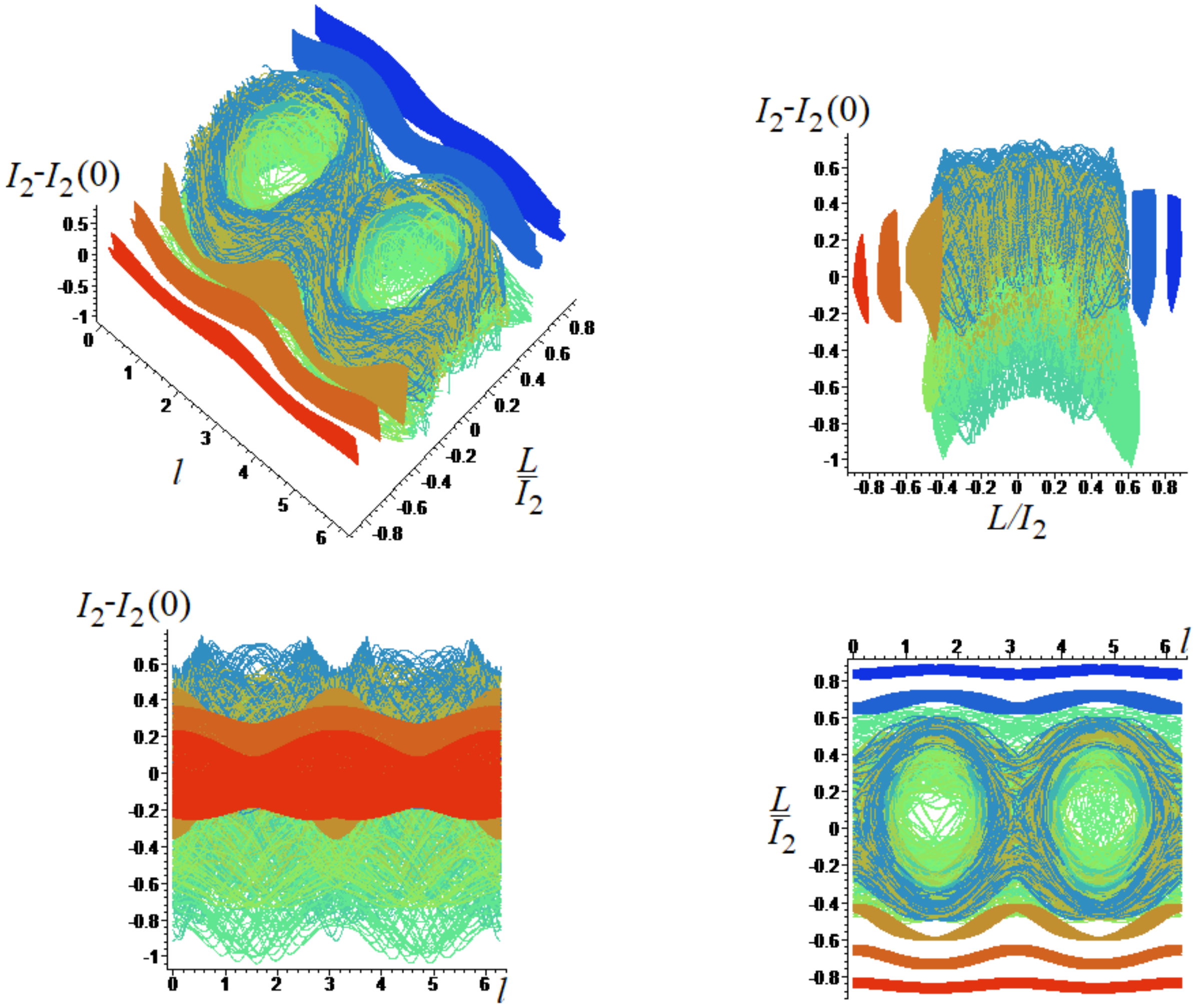


Fig.7. The full-dimensional phase portrait and its projections

From the fig.7 it is possible to see the mixing of phase trajectories with different energy values (colors from brown to blue) in the area of unperturbed separatrix (S_1-S_2-S'_1 at fig.6-a). This proves the splitting the unperturbed manifolds (stable and unstable) of the separatrix and the possibility of reciprocal phase trajectories penetrations from different phase regions through the initiated heteroclinic net. The same process can be observed from the Poincaré sections (fig.8), where clouds of points not-belonging to invariant curves are clearly depicted – such clouds, as it is known, are called as “chaotic layers”, and, certainly, the width of chaotic layer depends on the value of perturbations (tabl.1).

It will be fair to note that the majority of figures in this article corresponds to quite large values of perturbations and the angular momentum – it don’t influence in principle the fundamental nature of studied process, but it allows to fulfill more fast numerical calculations (with more high accuracy). To show the investigated process more closely to the real satellites motion (low orbits of the Earth, slow rotations) we can present separately the Poincaré section (fig.8-d) plotted with the small value of the gravitational parameter and angular momentum, that allows to see the same motion nature and chaotic phenomena.

In any case, all of the modelled motion modes can be considered as the different examples of the possible regular and chaotic attitude dynamical regimes of gyrostat-satellites in the central gravity field close to the conical precession regimes at the coincidence of the angular momentum with the direction of the central gravity vector. Here it is worth to remind, that the conical precession is one of the most useful regime of the attitude stabilization of prolate satellites – it defines the practical applicability of obtained results and demonstrated examples.

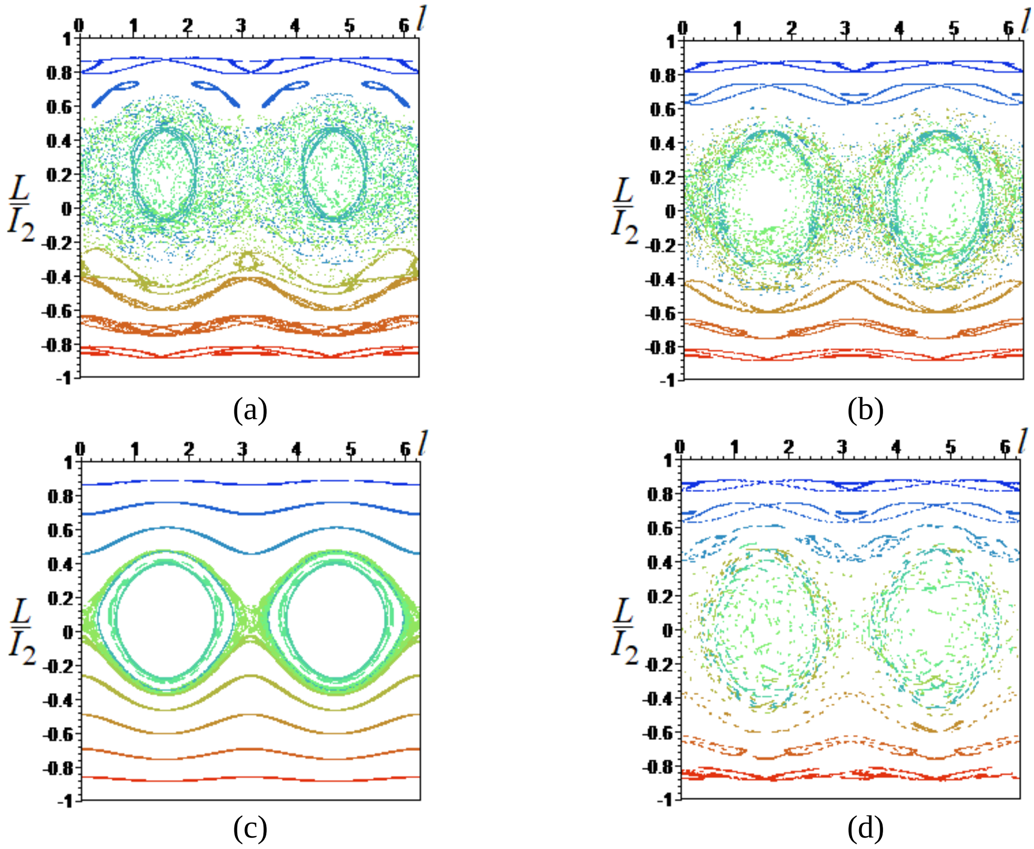


Fig.8. The Poincaré sections by the plane $I_2=I_2(0)=K$

Table 1 - *Numerical parameters*

Figures	μ [1/s^2]	ε	Initial values angular velocities [1/s]			Initial values for directional cosines			Angular momentums [kg·m^2/s]	
			p_0	q_0	r_0	γ_1	γ_2	γ_3	K	Δ
3-a, b	0.0046	0.086	0	0.254	0.149	0	0.954	0.3	4	0.155
3-c, d	0.046	0.863	0	0.254	0.149	0	0.954	0.3	4	0.155
4-a	0	0	-	-	-	-	-	-	4	0.550
4-b	0.053	0.994	-	-	-	-	-	-	4	0.550
4-c	0.075	1.406	-	-	-	-	-	-	4	0.550
4-d	0.150	2.813	-	-	-	-	-	-	4	0.550
4-e	0.210	3.938	-	-	-	-	-	-	4	0.550
4-f	0.500	9.375	-	-	-	-	-	-	4	0.550
5	0.046	0.863	0	0.266	0.019	0	0.997	0.073	4	0.155
6-a	0	0	-	-	-	0	0.954	0.3	4	0.155
6-b	0.046	0.863	-	-	-	0	0.954	0.3	4	0.155
7	0.046	0.863	-	-	-	-	-	-	4	0.155
8-a	0.055	1.031	-	-	-	-	-	-	4	0.550
8-b	0.046	0.863	-	-	-	-	-	-	4	0.155
8-c	0.0046	0.086	-	-	-	-	-	-	4	0.155
8-d	$5\cdot10^{-6}$	0.938	-	-	-	-	-	-	0.04	10^{-4}

6. Conclusion

The case of the gyrostat angular motion in the central gravity field at the condition of the collinearity of the angular momentum and the gravity field gradient was investigated. This case of the gyrostat dynamics is appropriate to the description of the angular motion of the axial gyrostat-satellite in the central gravity field at the realizations of important conical precessions regimes, when it is stabilized by the gravitational way along the local vertical direction. The corresponded analytical solution of the gyrostat/gyrostat-satellite angular motion was obtained in terms of elliptic functions. This solution can be characterized as the partial generalization of V.A. Stekloff case of the body motion.

The possibility of chaotization phenomena in the satellite angular motion close to the conical precession regime was investigated in the article. The dynamical chaos, as it was shown, arises due to the complicity of the dynamics near the heteroclinic separatrix, where the action of the torque from the central gravity forces makes the phase trajectory penetrate into different areas of the system phase space through the perturbed heteroclinic manifolds. This phenomenon should be taking into account at the developing space missions for prolate gyrostat-satellites using the gravitational attitude stabilization.

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