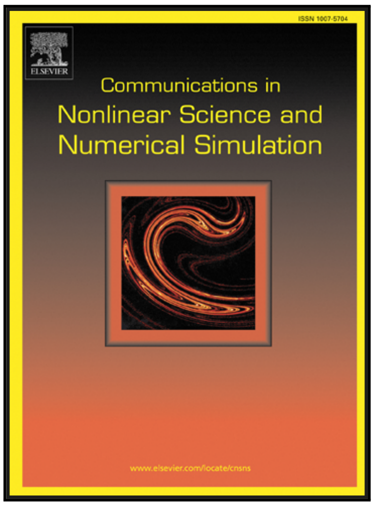

Heteroclinic chaos and its local suppression in attitude dynamics of an asymmetrical dual-spin spacecraft and gyrostat-satellites, The part I – main models and solutions

Anton V. Doroshin

PII: S1007-5704(15)00223-3
DOI: 10.1016/j.cnsns.2015.06.022
Reference: CNSNS 3592

To appear in: *Communications in Nonlinear Science and Numerical Simulation*

Received date: 18 February 2015
Revised date: 7 June 2015
Accepted date: 20 June 2015

Highlights

1. New generalized models are obtained for the dual-spin spacecraft with the constructional/mass-inertia asymmetry.

2. New action-angle-solutions for the heteroclinic case of the dynamics are found.

3. The Melnikow-Wiggins formalizm was used for the heteroclinic chaos analysis in the hamiltonian and non-Hamiltonian cases of the perturbations; the problems of the Melnikov-Wiggins methodology are indicated.

4. Chaos suppression techniques are considered and new heteroclinic chaos suppressing schemes are suggested.

Heteroclinic Chaos and Its Local Suppression in Attitude Dynamics of an Asymmetrical Dual-Spin Spacecraft and Gyrostat-Satellites

The Part I – Main models and solutions

Anton V. Doroshin, doran@inbox.ru

Space Engineering Department, Samara State Aerospace University (National Research University), SSAU, 34, Moskovskoe Shosse str., Samara, 443086, Russia

Abstract

The attitude dynamics of a dual-spin spacecraft (DSSC) and gyrostats-satellites is considered at the presence of the constructional asymmetry and at the action of the internal/external perturbations, including the friction between the coaxial DSSC bodies, electromotors' torques applied to the rotor-body from the platform-body by internal engines, the counterelectromotive forces/torques in internal engines, internal plyharmonic disturbances and external magnetic perturbations. New/modified mathematical models and dynamical systems are obtained for the investigation of the DSSC chaotic dynamics.

Keywords

Dual-Spin Spacecraft, Satellite, Gyrostat, Exact Heteroclinic Solutions, Action-Angle Variables, Chaotic Dynamics, Heteroclinic Net, Melnikov-Wiggins Methodology, Poincaré Map

Introduction

The detailed exploration of the attitude motion of spacecraft and satellites still remains one of the main problems of the space-flight-dynamics. As an important part of these problems we can indicate the motion analysis of a dual-spin spacecraft (DSSC) and gyrostats (also gyrostats-satellites). The DSSC/gyrostats are widely used types of spacecraft among communications satellites-systems and observing geostationary satellites. The coaxial dual-spin scheme usually is used for providing the attitude stabilization of the spacecraft by the partial-twist-method. The partial-twist-method implies the fast rotation only of the «rotor»-body. It allows to include into the «platform»-body exploratory equipment, and to implement space-missions tests without rotational perturbations.

It is possible to present concrete quite famous examples of DSSC, which were used in real space-programs. This is the well successful and long-continued project "Intelsat" (the Intelsat II launched in 1966), including 8-th generation of the geostationary communication satellites and Intelsat VI (1991), designed and built by Hughes Aircraft Company. The "Meteosat"-program (European Space Research Organization) used the DSSC-scheme; this program was initiated with Meteosat-1 in 1977 and operated with Meteosat-7 until 2007. The DSSC-construction was applied for the GEOTAIL mission launched in 1992 (and continued to operate in 2013) in the framework of the collaborative mission of Japan JAXA/ISAS and NASA, within the program "International Solar-Terrestrial Physics". The constructional DSSC-scheme of spacecraft (with the despun antenna) was selected in Chinese satellites DFH-2 (STW-3, 1988; STW-4, 1988; STW-55, 1990). The Galileo Jupiter-mission (1989) also used the dual-spin scheme. Certainly, we should underline the world's most-purchased commercial communication satellite's types Hughes/ Boeing HS-376/ BSS-376 (e.g. Satellite Business Systems with the implementation of projects SBS 1, 2, 3, 4, 5, 6 / HGS 5, etc.) – this spacecraft-scheme has the spun-section with the propulsion system and solar drums, and the despun-section with the payload and antennas. Very useful spacecraft models Hughes HS-381 (Leasat-project), HS-393 (JCSat-project), HS-389 (Intelsat-project) also have the dual-spin structure.

Especially interesting problems in the framework of the related modern research are the chaotic motion investigation and the chaos suppression in the DSSC dynamics at the action of external and internal perturbations. Here it is important to underline the problem of the detection of homo(hetero)clinic chaos in the DSSC phase space. The next step after the chaos detection is the task of its suppression/avoidance in the motion dynamics – there are multiple chaos control/suppressing techniques can be used; and most of them have the dissipative type of suppressing forces/torques, e.g. the external/internal resisting medium [Baozeng Y., Jiafang X.; El-Gohary A.; Iñarrea M.; Kuang J.L.; Leung A.Y.T.; Meechan P.A., Asokanthan S.F.; Zhou L.]. Also we ought to indicate corresponding related problems and results in the framework of nonlinear regular/chaotic dynamics [Anishchenko V.S., Astakhov V.V. at al.; Bainum P.M. at al.; Beletskiĭ V.V. at al.; Boccaletti S. at al.; Burov A.A.; Celletti A., Lhotka C.; Chaikin S.V.; Ge Z.-M., Lin T.-N.; Guckenheimer J.; Gutnik S.A.; Hall C.D. at al.; Holmes P. J.; Kinsey K. J.; Lin Yiing-Yuh, Wang Chin-Tzuo; Marsden J.E.; Meechan P.A., Asokanthan S.F.; Meng Y. et al.; Nazari M., Butcher E.A.; Pecora L.M. at al.; Rubanovskii V.M.; Sarychev V.A., Mirer S.A.; Seo at al.; Vera J.A.; Wiggins S.; Zhou at al.].

Chaotic phenomena in the DSSC dynamical system were considered in the wide spectrum of works [e.g., Aslanov, Bao-Zeng, Chen, Doroshin, El-Gohary, Ge, Hall, Holmes, Iñarrea, Lanchares, Leung, Kuang, Meechan, Neishtadt, Or, Peng, Shirazi, Tong, et al.] – these results will be surveyed and discussed in details below in corresponding sections of the paper. At the same time, the problem of the heteroclinic chaos analysis/avoidance/suppression in the DSSC dynamics is very broad in its conceptual and instrumental content; this problem is far from the solution taking into account various dynamical aspects and tasks' formulations. So, below we will consider the attitude dynamics of the asymmetric magnetized DSSC under the action of perturbations. Moreover, results of the paper will be connected with important practical applications, and will describe the asymmetric magnetized DSSC dynamics in the neighborhood of the cylindrical precession regime, which generalizes the cases of the free angular motion of the coaxial bodies systems, DSSC and gyrostats (gyrostat-satellites) at the presence of geometrical/ constructional/ inertia-mass asymmetries, and under the influence of the natural external/internal forces and torques, including the friction between the coaxial DSSC bodies, electromotors' torques applied to the rotor-body from the platform-body by internal engines, the counterelectromotive forces/torques in internal engines, internal plyharmonic and external magnetic perturbations.

Among the paper results will be presented new expanded and generalized mechanical/mathematical models, including canonical Hamiltonian forms of the perturbed dynamical systems for the asymmetric magnetized DSSC – these models will be collected in the first part of the paper (The Part I – The main models and solutions). Also in the second part of the paper (The Part II – The heteroclinic chaos investigation) the new analytical and numerical results of the application of the Melnikov's-Wiggins' methodology for the analysis of the heteroclinic chaos appearance/disappearance in the phase space of the DSSC will be obtained and demonstrated; and analytical and technical approaches for this chaos suppression/avoidance and corresponding suggestions will be given.

1. Mechanical and mathematical models

1.1. The geometrical and inertia-mass parameters of the asymmetrical DSSC

Let us consider the attitude dynamics of the DSSC at the presence of the constructional asymmetry and at the action of the internal/external perturbations, including the friction between the coaxial DSSC bodies, electromotors' torques applied to the rotor-body from the platform-body by internal engines, the counterelectromotive forces/torques in internal engines, internal plyharmonic disturbances and external magnetic perturbations. In this purpose we can present the mechanical model of the DSSC basing on the well-known coaxial-scheme construction with using the following coordinates systems (Fig.1):

$C_i\bar{x}_i\bar{y}_i\bar{z}_i$ – the connected principal central frame of references of the coaxial body #*i*, or, equivalently, the coordinates system connected to the main axes of the inertia ellipsoid of the coaxial body *i* (*i*=1, 2; #1 – the rotor-body, #2 – the coaxial platform-body), where C_i – is the mass center of the body *i*. Here it is needed to note that the coaxial bodies can have the complex structure with the corresponding complex construction of the real DSSC's rotor/platform, but from the mechanical point of view these bodies always have the natural inertial analogs as the inertia ellipsoids, which can be presented in the corresponding main axes. So, we will exactly consider the coaxial bodies as their inertia ellipsoids (Fig.1). In this frame the corresponding body has the principal inertia tensor with the diagonal form

$$(1.1)\qquad C_i\bar{x}_i\bar{y}_i\bar{z}_i: \quad \left[\bar{\mathbf{I}}_i\right] = diag\left(\bar{A}_i, \bar{B}_i, \bar{C}_i\right).$$

$C_i x_i y_i z_i$ – the connected central frame of references of the coaxial body #*i* (*i*=1, 2), which is corresponded to the repositioning of the system $C_i\bar{x}_i\bar{y}_i\bar{z}_i$ (in which the inertia tensor is principal) by the minimal final rotation around the point C_i into the position with the axis C_iz_i directed parallel to the axis of the relative rotation of the coaxial bodies (the constructional axis of the relative rotation of the DSSC: P_1P_2 at the fig.1). At this repositioning the inertia tensor loses the principal form at the corresponding orthogonal transformation with the rotational matrix $\mathbf{S}_i$; the corresponding inertia tensor is rewritten in the form with non-diagonal centrifugal inertia moments:

$$(1.2)\qquad C_i x_i y_i z_i: \quad \left[\mathbf{I}_i\right] = \mathbf{S}_i \cdot \left[\bar{\mathbf{I}}_i\right] \cdot \mathbf{S}_i^{-1}.$$

For example, we can consider this out-of-plane final orthogonal transformation as a pair of serial rotations: on the angle α_i around the "initial" direction of the axes $C_i\bar{x}_i$ and, after that, on the angle β_i around the "displaced" direction of the axes $C_i\tilde{\bar{y}}_i$. Then we will have the following structure of the final rotational matrix $\mathbf{S}_i$

$$(1.3)\qquad \mathbf{S}_i = \left[\boldsymbol{\beta}_i\right]\cdot\left[\boldsymbol{\alpha}_i\right] = \begin{bmatrix} \cos\beta_i & 0 & -\sin\beta_i \\ 0 & 1 & 0 \\ \sin\beta_i & 0 & \cos\beta_i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_i & \sin\alpha_i \\ 0 & -\sin\alpha_i & \cos\alpha_i \end{bmatrix};$$

$$\mathbf{S}_i = \begin{bmatrix} \cos\beta_i & \sin\beta_i\sin\alpha_i & -\sin\beta_i\cos\alpha_i \\ 0 & \cos\alpha_i & \sin\alpha_i \\ \sin\beta_i & -\sin\alpha_i\cos\beta_i & \cos\alpha_i\cos\beta_i \end{bmatrix};$$

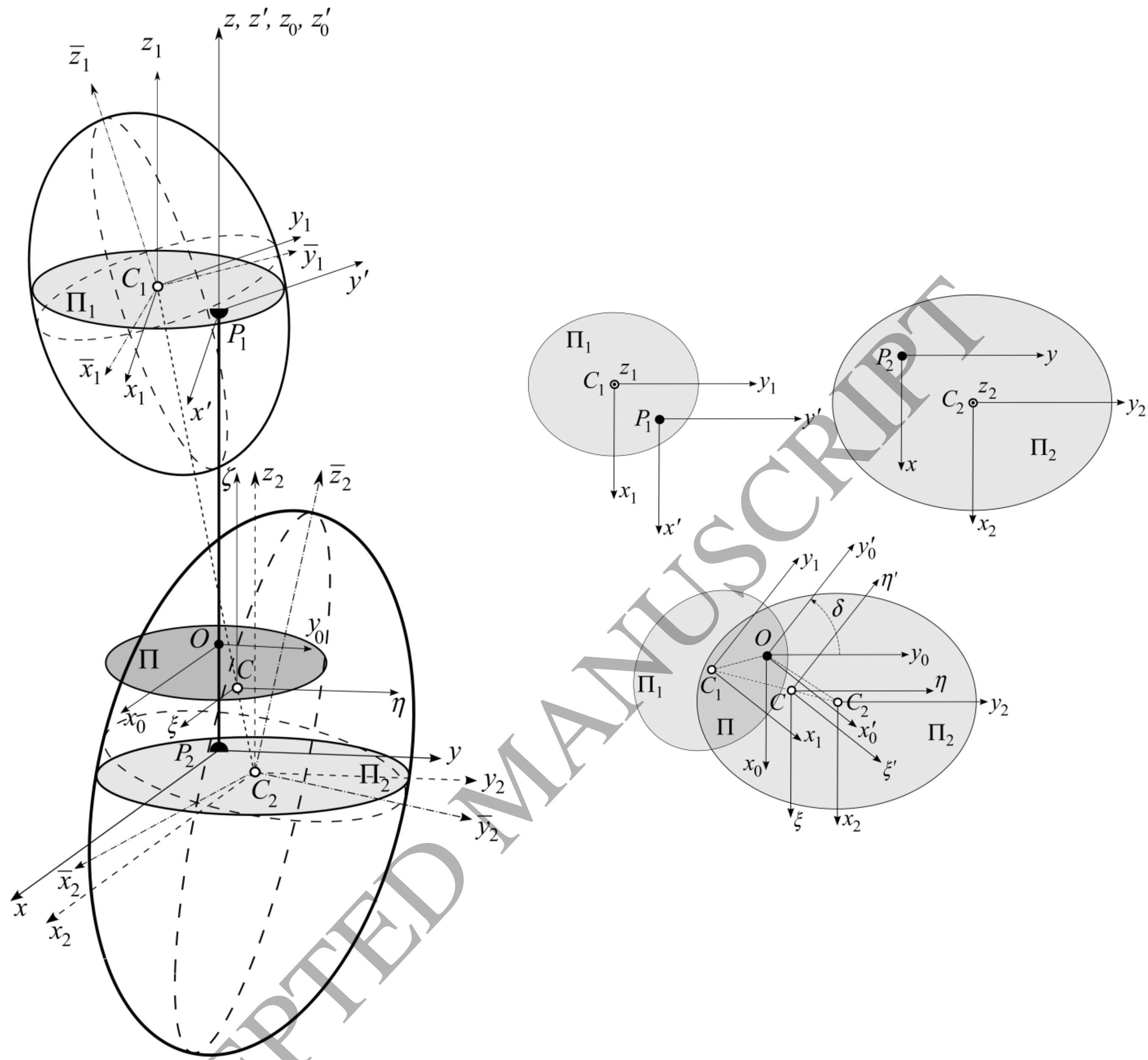

Fig.1. The geometrical DSSC-model and coordinates systems

The geometrical plane Π_i (i=1, 2) corresponds to the plane that is orthogonal to the axis of the coaxial relative rotation P_1P_2 and contains the body's mass center C_i ($C_i \in \Pi_i \perp P_1P_2$). Also it is important to note the coincidence of the body's connected plane $C_i x_i y_i$ with the plane Π_i. Moreover, the intersection of the plane $C_i x_i y_i$ with the relative rotation axis is located exactly in the point P_i: $P_i = \left(C_i x_i y_i \cap P_1P_2\right)$; and we can indicate/define the constant coordinates of the point P_i in the frame

$$(1.4) \quad C_i x_i y_i z_i: \; x_i\left(P_i\right) = -l_x^{(i)}; \; y_i\left(P_i\right) = -l_y^{(i)}; \; z_i\left(P_i\right) = 0.$$

Also we will use the connected frames P_2xyz (connected to the coaxial body-platform) and $P_1x'y'z'$ (connected to the coaxial body-rotor) with the origins P_i and with axes which are parallel to corresponding axes of systems $C_i x_i y_i z_i$: $P_1 x'y'z' \parallel C_1 x_1 y_1 z_1$; $P_2 xyz \parallel C_2 x_2 y_2 z_2$.

Now it is important to geometrically indicate/define the center of mass of the coaxial system C, and the geometrical point O on the coaxial axis P_1P_2 which corresponds to the "longitudinal level" of the system's mass center. It is clear that the relative rotation of the coaxial bodies with their displaced (relative to the rotation axes) mass centers results in the "geometrical motion" of the position of the system mass center relative to both coaxial bodies. But the relative coaxial rotations of the bodies is fulfilled in such way that their mass centers describes circles around the points P_i in the planes Π_i – therefore the system's mass center position always remains on the "constant connected" plane Π $\left(\Pi \parallel \Pi_i;\ O = \Pi \cap P_1P_2\right)$ at the geometrical motion relative to the coaxial bodies (Fig.1). Also it is geometrically clear that the "variable position" of the system's mass center relative coaxial bodies always represents the formal intersection of the line C_1C_2 with the plane Π $\left(C = C_1C_2 \cap \Pi\right)$. Of course, the real mechanical angular motion of the system in the inertial space will be fulfilled around the "motionless/fixed" position of the mass center, so we will consider the above mentioned "geometrical motion" of the mass center as the auxiliary geometrical interpretation.

After the definition of the geometrical positions of the points C and O we sequentially involve into consideration following auxiliary coordinates frames with the parallel axes (Fig.1): $Ox_0y_0z_0 \parallel P_2xyz$; $Ox_0'y_0'z_0' \parallel P_1x'y'z'$; $C\xi\eta\zeta \parallel Ox_0y_0z_0$; $C\xi'\eta'\zeta' \parallel Ox_0'y_0'z_0'$ (also we underline the coincidence of some axes: $z \equiv z' \equiv z_0 \equiv z_0'$; $\zeta \equiv \zeta'$).

Based on the indicated coordinates frames the important mass-inertial and kinematical parameters are defined – these parameters and their components are calculated in the connection to the specific frames:

$$\delta = \angle\left(Cx_0, Ox_0'\right) \equiv \angle\left(Oy_0, Oy_0'\right) \equiv \angle\left(C\xi, C\xi'\right)$$

(1.5) $$\left[\boldsymbol{\delta}\right] = \begin{bmatrix} \cos\delta & \sin\delta & 0 \\ -\sin\delta & \cos\delta & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \left[\boldsymbol{\delta}'\right] = \left[\boldsymbol{\delta}\right]^{-1} = \left[\boldsymbol{\delta}\right]^T$$

(1.6) $$\begin{cases} Ox_0y_0z_0: \ \overrightarrow{OC_2} = \boldsymbol{\rho}_{C_2} = \left[l_x^{(2)}, l_y^{(2)}, -OP_2\right]^T; \\ Ox_0'y_0'z_0': \ \overrightarrow{OC_1} = \boldsymbol{\rho}'_{C_1} = \left[l_x^{(1)}, l_y^{(1)}, OP_1\right]^T; \\ Ox_0y_0z_0: \ \overrightarrow{OC} = \boldsymbol{\rho}_C = \dfrac{1}{M_1+M_2}\left(M_2\boldsymbol{\rho}_{C_2} + M_1\left[\boldsymbol{\delta}'\right]\cdot\boldsymbol{\rho}'_{C_1}\right); \end{cases}$$

(1.7) $$\begin{cases} C\xi\eta\zeta: \ \overrightarrow{CO} = -\boldsymbol{\rho}_C; \\ C\xi\eta\zeta: \ \overrightarrow{CC_2} = \boldsymbol{\rho}_{C_2} - \boldsymbol{\rho}_C; \\ C\xi\eta\zeta: \ \overrightarrow{CC_1} = \left[\boldsymbol{\delta}'\right]\cdot\boldsymbol{\rho}'_{C_1} - \boldsymbol{\rho}_C; \\ C\xi'\eta'\zeta': \ \overrightarrow{CC_1} = \boldsymbol{\rho}'_{C_1} - \left[\boldsymbol{\delta}\right]\cdot\boldsymbol{\rho}_C; \end{cases}$$

where δ is the angle of the relative rotation of coaxial bodies; $[\boldsymbol{\delta}]$ – is the matrix of the coordinates systems transformations:

(1.8) $[\boldsymbol{\delta}]: Ox_0y_0z_0 \to Ox_0'y_0'z_0'$; $[\boldsymbol{\delta}]: C\xi\eta\zeta \to C\xi'\eta'\zeta'$;

$[\boldsymbol{\delta}']$ – is the back transformations matrix; $\boldsymbol{\rho}_{C_2}$ – is the vector of the mass center of the coaxial body-platform with its constant components presented in the frame $Ox_0y_0z_0$; $\boldsymbol{\rho}'_{C_1}$ – is the vector of the mass center of the coaxial body-rotor with its constant components presented in the frame $Ox_0'y_0'z_0'$; $\boldsymbol{\rho}_C$ – is the mass center of the complete coaxial system with its variable components in the frame $Ox_0y_0z_0$; M_i – is the mass of the coaxial body #*i*.

1.2. The kinematical and dynamical parameters of the angular motion

Let us describe the angular motion of the coaxial system in the inertial space assuming the motionless of the mass center of the coaxial system *C*. Then it is needed to involve the main inertial coordinate frame *CXYZ* (fig.1). So, we can distinguish the following aspects of the motion. It is clear that the DSSC coaxial system represents the mechanical system with four degree of freedom – three of them correspond to the angular motion of the platform around selected rotation pole (the mass center), and the fours degree of freedom describes the rotation of the coaxial rotor-body relative to the platform (the rotation of the rotor about coaxial axis P_1P_2 by the angle δ).

The angular motion of the coaxial bodies is described by the angular velocities, which can be written as vectors in the corresponding connected frames

$$C_2x_2y_2z_2: \quad \boldsymbol{\omega} = [p, q, r]^T; \qquad C_1x_1y_1z_1: \quad \boldsymbol{\omega}' = [p', q', r']^T$$

where $\boldsymbol{\omega}$ is the absolute angular velocity of the platform-body (around its mass center C_2) which is written in the body-platform's connected frame $C_2x_2y_2z_2$; $\boldsymbol{\omega}'$ is the absolute angular velocity of the rotor-body (around its mass center C_1) which is written in the body-rotor's connected frame $C_1x_1y_1z_1$. Taking into account indicated degrees of freedom, independence of the angular motion from the selected rotation pole and the interconnection of the coordinates frames (1.8) we can write the following expressions for the components of the angular velocities:

$$(1.9) \quad \begin{cases} p' = p\cos\delta + q\sin\delta; \\ q' = -p\sin\delta + q\cos\delta; \\ r' = r + \sigma; \end{cases}$$

$$(1.10) \quad \sigma = \dot{\delta}$$

σ – is the relative angular velocity of the rotor.

Now we can write main dynamical values for the bodies and for the system.

The body-rotor's linear momentum and the angular momentum (about the system mass center *C*) are

(1.11) $\mathbf{Q}_1 = M_1 \mathbf{v}_{C_1}^{abs}; \quad \mathbf{K}_C^{(1)} = \mathbf{K}_{C_1}^{(1)} + \overrightarrow{CC_1} \times \mathbf{Q}_1$

where the absolute velocity of the mass center of the rotor-body can be presented in projections on the axes $C\xi'\eta'\zeta'$:

$$\mathbf{v}_{C_1}^{abs} = \mathbf{v}_{P_1}^{abs} + \boldsymbol{\omega}_{C_1P_1} \times \overrightarrow{P_1C_1} = \left[[\boldsymbol{\delta}] \cdot \left(\boldsymbol{\omega} \times \overrightarrow{CP_1} \right) + \boldsymbol{\omega}' \times \overrightarrow{P_1C_1} \right] =$$

(1.12) $$= \left[[\boldsymbol{\delta}] \cdot \left(\boldsymbol{\omega} \times \left(\overrightarrow{CO} + \overrightarrow{OP_1} \right) \right) + \boldsymbol{\omega}' \times \overrightarrow{P_1C_1} \right] =$$

$$= \left[[\boldsymbol{\delta}] \cdot \left(\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \left((-\boldsymbol{\rho}_C) + \begin{bmatrix} 0 \\ 0 \\ OP_1 \end{bmatrix} \right) \right) + \begin{bmatrix} p' \\ q' \\ r' \end{bmatrix} \times \begin{bmatrix} l_x^{(1)} \\ l_y^{(1)} \\ 0 \end{bmatrix} \right]$$

Then the angular momentum of the rotor relative to the point C in projections on the axes $C\xi'\eta'\zeta'$ takes the form

(1.13) $$\mathbf{K}_{C\xi'\eta'\zeta'}^{(1)} = [\mathbf{I}_1] \cdot \begin{bmatrix} p' \\ q' \\ r' \end{bmatrix} + \left(\boldsymbol{\rho}'_{C_1} - [\boldsymbol{\delta}] \cdot \boldsymbol{\rho}_C \right) \times M_1 \left[[\boldsymbol{\delta}] \cdot \left(\begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \left((-\boldsymbol{\rho}_C) + \begin{bmatrix} 0 \\ 0 \\ OP_1 \end{bmatrix} \right) \right) + \begin{bmatrix} p' \\ q' \\ r' \end{bmatrix} \times \begin{bmatrix} l_x^{(1)} \\ l_y^{(1)} \\ 0 \end{bmatrix} \right]$$

The kinetic energy of the rotor is

(1.14) $$T_1 = T_{C_1} + T_{C_1}^{(r)} = \frac{1}{2} M_1 \left(\mathbf{v}_{C_1}^{abs} \right)^2 + \frac{1}{2} [p', q', r'] [\mathbf{I}_1] \begin{bmatrix} p' \\ q' \\ r' \end{bmatrix}$$

The body-platform’s linear momentum and the angular momentum (about the system mass center C) are

(1.15) $\mathbf{Q}_2 = M_2 \mathbf{v}_{C_2}^{abs}; \quad \mathbf{K}_C^{(2)} = \mathbf{K}_{C_2}^{(2)} + \overrightarrow{CC_2} \times \mathbf{Q}_2$

where the absolute velocity of the mass center of the platform-body can be presented in projections on the axes $C\xi\eta\zeta$:

(1.16) $\mathbf{v}_{C_2}^{abs} = \boldsymbol{\omega} \times \overrightarrow{CC_2} = \boldsymbol{\omega} \times \left(\boldsymbol{\rho}_{C_2} - \boldsymbol{\rho}_C \right)$

Then the angular momentum of the platform about the point C in projections on the axes $C\xi\eta\zeta$ takes the form

(1.17) $$\mathbf{K}_{C\xi\eta\zeta}^{(2)} = [\mathbf{I}_2] \cdot \begin{bmatrix} p \\ q \\ r \end{bmatrix} + \left(\boldsymbol{\rho}_{C_2} - \boldsymbol{\rho}_C \right) \times \left[M_2 \boldsymbol{\omega} \times \left(\boldsymbol{\rho}_{C_2} - \boldsymbol{\rho}_C \right) \right]$$

The kinetic energy of the platform is

(1.18) $$T_2 = T_{C_2} + T_{C_2}^{(r)} = \frac{1}{2} M_2 \left(\mathbf{v}_{C_2}^{abs} \right)^2 + \frac{1}{2} [p, q, r] [\mathbf{I}_2] \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

The main dynamical equations can be written using the well-known angular momentum theorem (1.19), and the methodology of the evaluation of the absolute derivative through local derivatives (in moving frames $C\xi\eta\zeta$ and $C\xi'\eta'\zeta'$) with the final projection (1.20) of the vectors into the main moving frame $C\xi\eta\zeta$ [Aslanov, Doroshin (2002)]:

(1.19) $$\frac{d}{dt} \mathbf{K}_C = \mathbf{M}_C^{(e)}$$

(1.20) $$[\boldsymbol{\delta}] \left[\frac{d}{dt} \mathbf{K}_{C\varsigma'\eta'\zeta'}^{(1)} + \boldsymbol{\omega}' \times \mathbf{K}_{C\varsigma'\eta'\zeta'}^{(1)} \right] + \left[\frac{d}{dt} \mathbf{K}_{C\varsigma\eta\zeta}^{(2)} + \boldsymbol{\omega} \times \mathbf{K}_{C\varsigma\eta\zeta}^{(2)} \right] = \mathbf{M}_{C\varsigma\eta\zeta}^{(e)}$$

where $\mathbf{M}_{C\varsigma\eta\zeta}^{(e)}$ is the vector of external torques, and all of the kinematical parameters are rewritten through the main components of the angular velocity of the platform (p,q,r) and the relative angular velocity of the rotor σ using (1.5)-(1.9).

The relative motion equation can be constructed basing on changing the "longitudinal" components of the angular momentum of the rotor at the action of the external torque $M_{C\zeta',1}^{(e)}$ (applied to the rotor-body) and the torque from the platform-body $M_\delta^{(i)}$ (e.g. internal friction between coaxial bodies):

(1.21) $$\frac{d}{dt} K_{C\zeta'}^{(1)} = M_{C\zeta',1}^{(e)} + M_\delta^{(i)}$$

The kinematical equations are correspond to the well-known Euler's equations for Euler's angles (the precession ψ, the nutation θ, and the intrinsic rotation φ, with the addition of the relative angle δ), which define the attitude of the main moving frame $C\xi\eta\zeta$ relative to the inertial axes *CXYZ* (fig.3):

(1.22) $$\begin{cases} \dot{\psi} = \dfrac{1}{\sin\theta} \left(p \sin\varphi + q \cos\varphi \right); & \dot{\theta} = p \cos\varphi - q \sin\varphi; \\ \dot{\varphi} = r - \operatorname{ctg}\theta \left(p \sin\varphi + q \cos\varphi \right); & \dot{\delta} = \sigma \end{cases}$$

1.3. The Hamiltonian form of equations at the presence of small dynamical asymmetry in the coaxial system

1.3.1. Perturbing factors

Firstly, let us describe the main factors of the possible asymmetry in the coaxial system. This asymmetry results in the perturbed dynamics of the system motion and also it can initiate the complex irregular dynamical regimes including heteroclinic chaos.

1. ***The asymmetry factor #1*** – the smallness (ε_I) of the "diagonal" asymmetry of the inertia tensors.

This type of the small asymmetry factor corresponds to the smallness of values of "rotation angles" (α_i, β_i) of the inertia tensors' matrix with the recalculation in the first approximation

$$\varepsilon_{\mathbf{I}} = \sup\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$$

$$\begin{aligned} &\alpha_1 = \varepsilon_{\mathbf{I}}\bar{\alpha}_1, \quad \alpha_2 = \varepsilon_{\mathbf{I}}\bar{\alpha}_2, \\ &\beta_1 = \varepsilon_{\mathbf{I}}\bar{\beta}_1, \quad \beta_2 = \varepsilon_{\mathbf{I}}\bar{\beta}_2, \\ &\bar{\alpha}_i, \bar{\beta}_i = \text{const} \le 1 \quad (i = 1, 2) \end{aligned}$$

(1.23) $$\mathbf{S}_i = |\alpha_i, \beta_i \sim \varepsilon| = \begin{bmatrix} 1 & 0 & -\beta_i \\ 0 & 1 & \alpha_i \\ \beta_i & -\alpha_i & 1 \end{bmatrix}$$

Then the tensor of inertia takes the decomposed form (in the linear approximation and with the selection of the small part)

(1.24)
$$\begin{aligned} [\mathbf{I}_i] &= \begin{bmatrix} 1 & 0 & -\beta_i \\ 0 & 1 & \alpha_i \\ \beta_i & -\alpha_i & 1 \end{bmatrix} \begin{bmatrix} \bar{A}_i & 0 & 0 \\ 0 & \bar{B}_i & 0 \\ 0 & 0 & \bar{C}_i \end{bmatrix} \begin{bmatrix} 1 & 0 & \beta_i \\ 0 & 1 & -\alpha_i \\ -\beta_i & \alpha_i & 1 \end{bmatrix} = \\ &= \begin{bmatrix} \bar{A}_i & 0 & \varepsilon_{\mathbf{I}}\bar{\beta}_i(\bar{A}_i - \bar{C}_i) \\ 0 & \bar{B}_i & \varepsilon_{\mathbf{I}}\bar{\alpha}_i(\bar{C}_i - \bar{B}_i) \\ \varepsilon_{\mathbf{I}}\bar{\beta}_i(\bar{A}_i - \bar{C}_i) & \varepsilon_{\mathbf{I}}\bar{\alpha}_i(\bar{C}_i - \bar{B}_i) & \bar{C}_i \end{bmatrix} = \\ &= \begin{bmatrix} \bar{A}_i & 0 & 0 \\ 0 & \bar{B}_i & 0 \\ 0 & 0 & \bar{C}_i \end{bmatrix} + \varepsilon_{\mathbf{I}} \begin{bmatrix} 0 & 0 & \bar{\beta}_i(\bar{A}_i - \bar{C}_i) \\ 0 & 0 & \bar{\alpha}_i(\bar{C}_i - \bar{B}_i) \\ \bar{\beta}_i(\bar{A}_i - \bar{C}_i) & \bar{\alpha}_i(\bar{C}_i - \bar{B}_i) & 0 \end{bmatrix} \end{aligned}$$

2. ***The asymmetry factor #2*** – the smallness (ε_B) of difference of the equatorial inertia moments of the coaxial rotor.

This type of the small asymmetry factor corresponds to the smallness of the dimensionless value of the relation

$$\varepsilon_B = \frac{\bar{A}_1 - \bar{B}_1}{\bar{A}_1}; \qquad \bar{B}_1 = \bar{A}_1(1 - \varepsilon_B)$$

In addition to the factor #1 we obtain the form for the inertia tensor of the rotor with the small dynamical asymmetry in the compliance with the factor #2 (eliminating the terms proportional to the product of factors $\varepsilon_I \varepsilon_B$)

(1.25)
$$\begin{aligned} [\mathbf{I}_1] &= \begin{bmatrix} \bar{A}_1 & 0 & 0 \\ 0 & \bar{A}_1(1-\varepsilon_B) & 0 \\ 0 & 0 & \bar{C}_1 \end{bmatrix} + \varepsilon_{\mathbf{I}} \begin{bmatrix} 0 & 0 & \bar{\beta}_1(\bar{A}_1 - \bar{C}_1) \\ 0 & 0 & \bar{\alpha}_1(\bar{C}_1 - \bar{A}_1(1-\varepsilon_B)) \\ \bar{\beta}_1(\bar{A}_1 - \bar{C}_1) & \bar{\alpha}_1(\bar{C}_1 - \bar{A}_1(1-\varepsilon_B)) & 0 \end{bmatrix} = \\ &= \begin{bmatrix} \bar{A}_1 & 0 & 0 \\ 0 & \bar{A}_1 & 0 \\ 0 & 0 & \bar{C}_1 \end{bmatrix} + (\bar{A}_1 - \bar{C}_1) \begin{bmatrix} 0 & 0 & \varepsilon_{\mathbf{I}}\bar{\beta}_1 \\ 0 & \dfrac{-\varepsilon_B \bar{A}_1}{\bar{A}_1 - \bar{C}_1} & -\varepsilon_{\mathbf{I}}\bar{\alpha}_1 \\ \varepsilon_{\mathbf{I}}\bar{\beta}_1 & -\varepsilon_{\mathbf{I}}\bar{\alpha}_1 & 0 \end{bmatrix} \end{aligned}$$

3. ***The asymmetry factor #3*** – the smallness (ε_l) of the linear displacement of the coaxial bodies axes from the common rotation axis.

   The factor #3 can be described by the following relation

$$\varepsilon_l = \frac{l}{P_1P_2}, \quad l = \sup\left\{l_x^{(1)}, l_y^{(1)}, l_x^{(2)}, l_y^{(2)}\right\}$$

$$l_x^{(1)} = \varepsilon_l l_{1x}; \;\; l_y^{(1)} = \varepsilon_l l_{1y}; \;\; l_x^{(2)} = \varepsilon_l l_{2x}; \;\; l_y^{(2)} = \varepsilon_l l_{2y};$$
$$l_{ij} = \text{const} \le 1 \quad \left(i = 1,2;\; j = x, y\right)$$

So, finally, we can involve the main small dimensionless parameter by the selection of the biggest value from considered factors:

(1.26) $$\varepsilon = \max\left\{\varepsilon_{\mathbf{I}}, \varepsilon_B, \varepsilon_l\right\}; \qquad \varepsilon_{\mathbf{I}} = e_{\mathbf{I}}\varepsilon; \quad \varepsilon_B = e_B\varepsilon; \quad \varepsilon_l = e_l\varepsilon$$

where e_I, e_B, e_l =const≤1.

1.3.2. Perturbed dynamical parameters

Now we can simplify the main geometrical, inertial, kinematical and dynamical parameters taking into account only linear parts of corresponding series (~ε).

(1.27) $$\begin{cases} Ox_0y_0z_0: \;\; \boldsymbol{\rho}_{C_2} = \varepsilon e_l\left[l_{2x}, l_{2y}, -OP_2\right]^T; \\ Ox_0'y_0'z_0': \;\; \boldsymbol{\rho}'_{C_1} = \varepsilon e_l\left[l_{1x}, l_{1y}, OP_1\right]^T; \\ Ox_0y_0z_0: \;\; \boldsymbol{\rho}_C = \dfrac{\varepsilon e_l}{M}\begin{bmatrix} M_2 l_{2x} + M_1\left(l_{1x}\cos\delta - l_{1y}\sin\delta\right) \\ M_2 l_{2y} + M_1\left(l_{1x}\sin\delta + l_{1y}\cos\delta\right) \\ 0 \end{bmatrix}; \\ C\xi\eta\zeta: \;\; \overrightarrow{CC_2} = \varepsilon e_l\begin{bmatrix} l_{2x}\left(1 - \dfrac{M_2}{M}\right) - \dfrac{M_1}{M}\left(l_{1x}\cos\delta - l_{1y}\sin\delta\right) \\ l_{2y}\left(1 - \dfrac{M_2}{M}\right) - \dfrac{M_1}{M}\left(l_{1x}\sin\delta + l_{1y}\cos\delta\right) \\ -OP_2 \end{bmatrix}; \end{cases}$$

where $M = M_1 + M_2$.

(1.28) $$\mathbf{v}_{C_1}^{abs} = \begin{bmatrix} OP_1 q' - r\left(P_y\cos\delta - P_x\sin\delta\right) - \varepsilon e_l l_{1y} r' \\ -OP_1 p' + r\left(P_x\cos\delta - P_y\sin\delta\right) + \varepsilon e_l l_{1x} r' \\ P_y p - P_x q + \varepsilon e_l l_{1y} p' - \varepsilon e_l l_{1x} q' \end{bmatrix};$$

$$(1.29)\quad \mathbf{v}_{C_2}^{abs} = \boldsymbol{\omega} \times \overrightarrow{CC_2} = \begin{bmatrix} -OP_2 q - \left(P_y + \varepsilon e_l l_{2y}\right) r \\ OP_2 p + \left(P_x + \varepsilon e_l l_{2x}\right) r \\ \left(P_y + \varepsilon e_l l_{2y}\right) p - \left(P_x + \varepsilon e_l l_{2x}\right) q \end{bmatrix};$$

$$P_x = \frac{\varepsilon e_l}{M}\left(-M_2 l_{2x} - M_1\left[l_{1x}\cos\delta - l_{1y}\sin\delta\right]\right);$$
$$P_y = \frac{\varepsilon e_l}{M}\left(-M_2 l_{2y} - M_1\left[l_{1x}\sin\delta + l_{1y}\cos\delta\right]\right);$$

The angular momentum of the platform-body is

$$\mathbf{K}_{C\xi\eta\zeta}^{(2)} = \begin{bmatrix} \left(\bar{A}_2 + M_2 \cdot OP_2^2\right) p \\ \left(\bar{B}_2 + M_2 \cdot OP_2^2\right) q \\ \bar{C}_2 r \end{bmatrix} + \varepsilon e_I \begin{bmatrix} \bar{\beta}_2(\bar{A}_2 - \bar{C}_2) r \\ \bar{\alpha}_2(\bar{C}_2 - \bar{B}_2) r \\ \bar{\beta}_2(\bar{A}_2 - \bar{C}_2) p + \bar{\alpha}_2(\bar{C}_2 - \bar{B}_2) q \end{bmatrix} +$$
$$+\varepsilon e_l \frac{M_2 M_1 OP_2}{M} \begin{bmatrix} \left(l_{2x} - \left[l_{1x}\cos\delta - l_{1y}\sin\delta\right]\right) r \\ \left(l_{2y} - \left[l_{1x}\sin\delta + l_{1y}\cos\delta\right]\right) r \\ \left(l_{2x} - \left[l_{1x}\cos\delta - l_{1y}\sin\delta\right]\right) p + \left(l_{2y} - \left[l_{1x}\sin\delta + l_{1y}\cos\delta\right]\right) q \end{bmatrix}$$

The angular momentum of the rotor-body is:

$$\mathbf{K}_{C\xi'\eta'\zeta'}^{(1)} = \begin{bmatrix} \left(\bar{A}_1 + M_1 OP_1^2\right)\left(p\cos\delta + q\sin\delta\right) \\ \left(\bar{A}_1 + M_1 OP_1^2\right)\left(q\cos\delta - p\sin\delta\right) \\ \bar{C}_1\left(r + \sigma\right) \end{bmatrix} +$$
$$+\varepsilon e_{\mathbf{I}}\left(\bar{A}_1 - \bar{C}_1\right) \begin{bmatrix} \bar{\beta}_1\left(r + \sigma\right) \\ -\bar{\alpha}_1\left(r + \sigma\right) \\ p\left(\bar{\beta}_1\cos\delta + \bar{\alpha}_1\sin\delta\right) + q\left(\bar{\beta}_1\sin\delta - \bar{\alpha}_1\cos\delta\right) \end{bmatrix} -$$
$$-\varepsilon e_B \bar{A}_1 \begin{bmatrix} 0 \\ \left(-p\sin\delta + q\cos\delta\right) \\ 0 \end{bmatrix} +$$
$$+\varepsilon e_l \frac{M_2 M_1 OP_1}{M} \begin{bmatrix} r\left(l_{2y}\sin\delta + l_{2x}\cos\delta - l_{1x}\right) - l_{1x}\sigma \\ r\left(l_{2y}\cos\delta - l_{2x}\sin\delta - l_{1y}\right) - l_{1y}\sigma \\ p\left(l_{2x} + l_{1y}\sin\delta - l_{1x}\cos\delta\right) + q\left(l_{2y} - l_{1x}\sin\delta - l_{1y}\cos\delta\right) \end{bmatrix}$$

So, the system angular momentum can be written in the main connected platform's frame $C\xi\eta\zeta$

$$\mathbf{K}_{C\xi\eta\zeta} = \mathbf{K}_{C\xi\eta\zeta}^{(2)} + \left[\boldsymbol{\delta}'\right] \cdot \mathbf{K}_{C\xi'\eta'\zeta'}^{(1)}$$

Now we have the expressions for the angular momentum of the coaxial system with the complex asymmetry, therefore, we can write expressions for the Serret-Andoyer variables [Serret (1866);

Andoyer (1923); Deprit (1967)] $\{l, L\}, \{\varphi_2, G\}, \{\varphi_3, H\}, \{\delta, \Delta\}$ (also known as the Andoyer-Deprit variables [Arkhangelskiĭ (1977); Ivin (1985)]). These variables describe the position of the angular momentum vector in the inertial frame of references and also in the connected frame (fig.2). The Serret-Andoyer variables can be involved by the classical way of correspondences finding between the variables and the angular momentum's components:

$$\begin{aligned} &L = K_{C\zeta};\ G = |\mathbf{K}|;\ H = K_{CZ};\ \Delta = K_{C\zeta}^{(1)} = K_{C\zeta'}^{(1)}; \\ &\sqrt{G^2 - L^2}\sin l = K_{C\xi};\ \sqrt{G^2 - L^2}\cos l = K_{C\eta}; \end{aligned} \tag{1.30}$$

After substituting the dependencies for the angular momentum components into (1.30) the exact expressions for the angular velocities in terms of the Serret-Andoyer variables follow as the formal solution of the linear algebraic equations relatively components $\{p, q, r, \sigma\}$ – these expressions called as the "conjunctional" expressions:

$$\begin{cases} p = \hat{p}(l, L, \delta, \Delta, G) = \hat{p}_0(l, L, \delta, \Delta, G) + \varepsilon \hat{p}_1(l, L, \delta, \Delta, G); \\ q = \hat{q}(l, L, \delta, \Delta, G) = \hat{q}_0(l, L, \delta, \Delta, G) + \varepsilon \hat{q}_1(l, L, \delta, \Delta, G); \\ r = \hat{r}(l, L, \delta, \Delta, G) = \hat{r}_0(l, L, \delta, \Delta, G) + \varepsilon \hat{r}_1(l, L, \delta, \Delta, G); \\ \sigma = \hat{\sigma}(l, L, \delta, \Delta, G) = \hat{\sigma}_0(l, L, \delta, \Delta, G) + \varepsilon \hat{\sigma}_1(l, L, \delta, \Delta, G). \end{cases} \tag{1.31}$$

It is possible to allocate the "unperturbed" $(\hat{p}_0, \hat{q}_0, \hat{r}_0, \hat{\sigma}_0)$ and "perturbed" $(\hat{p}_1, \hat{q}_1, \hat{r}_1, \hat{\sigma}_1)$ components linked with the concrete asymmetry factor:

$$\hat{p}_0 = \frac{1}{A}\sqrt{G^2 - L^2}\sin l;\quad \hat{q}_0 = \frac{1}{B}\sqrt{G^2 - L^2}\cos l;\quad \hat{r}_0 = \frac{L - \Delta}{\bar{C}_2};\qquad \hat{\sigma}_0 = \frac{(\bar{C}_1 + \bar{C}_2)\Delta - \bar{C}_1 L}{\bar{C}_1 \bar{C}_2}; \tag{1.32}$$

$$\hat{p}_1 = e_{\mathbf{I}}\hat{p}_{\mathbf{I}} + e_l \hat{p}_l + e_B \hat{p}_B;\ \hat{q}_1 = e_{\mathbf{I}}\hat{q}_{\mathbf{I}} + e_l \hat{q}_l + e_B \hat{q}_B;\quad \hat{r}_1 = e_{\mathbf{I}}\hat{r}_{\mathbf{I}} + e_l \hat{r}_l + e_B \hat{r}_B;\quad \hat{\sigma}_1 = e_{\mathbf{I}}\hat{\sigma}_{\mathbf{I}} + e_l \hat{\sigma}_l + e_B \hat{\sigma}_B; \tag{1.33}$$

where (considering that $(M_2 OP_2 - M_1 OP_1)/M = -\zeta_C = 0$)

$$\hat{p}_{\mathbf{I}} = \frac{1}{A\bar{C}_1\bar{C}_2}\left(\bar{\beta}_2\bar{C}_1(\bar{A}_2 - \bar{C}_2)(\Delta - L) - [\bar{\beta}_1\cos\delta + \bar{\alpha}_1\sin\delta]\bar{C}_2(\bar{A}_1 - \bar{C}_1)\Delta\right);$$

$$\hat{p}_B = \bar{A}_1\sqrt{G^2 - L^2}\left(\frac{1}{A^2}\sin l\sin^2\delta - \frac{1}{AB}\cos l\cos\delta\sin\delta\right);$$

$$\hat{p}_l = \frac{1}{A\bar{C}_1\bar{C}_2}\left((\Delta - L)\bar{C}_1 M_2 OP_2 l_{2x} + \Delta\bar{C}_2 M_1 OP_1[l_{1x}\cos\delta - l_{1y}\sin\delta]\right);$$

$$\hat{q}_{\mathbf{I}} = \frac{1}{B\bar{C}_1\bar{C}_2}\left(\bar{\alpha}_2\bar{C}_1(\bar{C}_2 - \bar{B}_2)(\Delta - L) - [\bar{\beta}_1\sin\delta - \bar{\alpha}_1\cos\delta]\bar{C}_2(\bar{A}_1 - \bar{C}_1)\Delta\right);$$

$$\hat{q}_B = \bar{A}_1\sqrt{G^2 - L^2}\left(\frac{1}{B^2}\cos l\cos^2\delta - \frac{1}{AB}\sin l\cos\delta\sin\delta\right);$$

$$\hat{q}_l = \frac{1}{B\bar{C}_1\bar{C}_2}\left((\Delta - L)\bar{C}_1 M_2 OP_2 l_{2y} + \Delta\bar{C}_2 M_1 OP_1[l_{1y}\cos\delta + l_{1x}\sin\delta]\right);$$

$$\hat{r}_{\mathbf{I}} = -\frac{\sqrt{G^2 - L^2}}{\bar{C}_2}\left(\frac{\bar{\beta}_2}{A}(\bar{A}_2 - \bar{C}_2)\sin l + \frac{\bar{\alpha}_2}{B}(\bar{C}_2 - \bar{B}_2)\cos l\right);$$

$$\hat{r}_B = 0;$$

$$\hat{r}_l = \frac{M_1\sqrt{G^2-L^2}}{M\bar{C}_2}\left(\frac{\sin l}{A}\left(M_1OP_1\left[l_{1x}\cos\delta - l_{1y}\sin\delta\right] - M_2OP_2l_{2x}\right) - \right.$$
$$\left. -\frac{\cos l}{B}\left(M_1OP_1\left[l_{1y}\cos\delta + l_{1x}\sin\delta\right] - M_2OP_2l_{2y}\right)\right);$$
$$\hat{\sigma}_{\mathrm{I}} = \sqrt{G^2-L^2}\left(\left[\frac{\left(\bar{C}_1-\bar{A}_1\right)}{B\bar{C}_1}\bar{\beta}_1\sin\delta - \frac{\left(\bar{C}_1-\bar{A}_1\right)}{B\bar{C}_1}\bar{\alpha}_1\cos\delta + \frac{\bar{\alpha}_2\left(\bar{C}_2-\bar{B}_2\right)}{B\bar{C}_2}\right]\cos l + \right.$$
$$\left. +\left[\frac{\left(\bar{C}_1-\bar{A}_1\right)}{A\bar{C}_1}\bar{\alpha}_1\sin\delta + \frac{\left(\bar{C}_1-\bar{A}_1\right)}{A\bar{C}_1}\bar{\beta}_1\cos\delta + \frac{\bar{\beta}_2\left(\bar{A}_2-\bar{C}_2\right)}{A\bar{C}_2}\right]\sin l\right);$$
$$\hat{\sigma}_B = 0;$$
$$\hat{\sigma}_l = \frac{M_1M_2\left(OP_1\bar{C}_2 - OP_2\bar{C}_1\right)\sqrt{G^2-L^2}}{M\bar{C}_1\bar{C}_2}\left(\frac{\cos l}{B}\left(l_{1x}\sin\delta + l_{1y}\cos\delta - l_{2y}\right) + \frac{\sin l}{A}\left(l_{1x}\cos\delta - l_{1y}\sin\delta - l_{2x}\right)\right)$$

with the flowing notations for the inertia moments

$$A = M_2OP_2^2 + \bar{A}_2 + M_1OP_1^2 + \bar{A}_1;\; B = M_2OP_2^2 + \bar{B}_2 + M_1OP_1^2 + \bar{A}_1;\; C = \bar{C}_1 + \bar{C}_2$$

So, now we can write the kinetic energy form taking into account only linear parts of the expansions by the small parameter

$$T = T_{platform} + T_{rotor}$$

$$T_{platform} = \frac{1}{2}\left(\left[\bar{A}_2 + M_2OP_2^2\right]p^2 + \left[\bar{B}_2 + M_2OP_2^2\right]q^2 + \bar{C}_2r^2\right) +$$
$$+\varepsilon e_I\left[\bar{\beta}_2(\bar{A}_2-\bar{C}_2)rp + \bar{\alpha}_2(\bar{C}_2-\bar{B}_2)rq\right] +$$
$$+\varepsilon e_l OP_2M_2\left[\left(l_{2y}\left[1-\frac{M_2}{M}\right] - \frac{M_1}{M}\left[l_{1x}\sin\delta + l_{1y}\cos\delta\right]\right)qr + \left(l_{2x}\left[1-\frac{M_2}{M}\right] - \frac{M_1}{M}\left[l_{1x}\cos\delta - l_{1y}\sin\delta\right]\right)pr\right] +$$
$$+O\left(\varepsilon^2\right);$$

$$T_{rotor} = \frac{1}{2}\left(\left(\bar{A}_1 + M_1OP_1^2\right)\left(q^2+p^2\right) + \bar{C}_1\left(r+\sigma\right)^2\right) - \frac{1}{2}\varepsilon e_B\bar{A}_1\left(-p\sin\delta + q\cos\delta\right)^2 +$$
$$+\varepsilon e_I\left(\bar{A}_1-\bar{C}_1\right)\left(r+\sigma\right)\left(\bar{\beta}_1\left[p\cos\delta + q\sin\delta\right] - \bar{\alpha}_1\left[-p\sin\delta + q\cos\delta\right]\right) +$$
$$+\varepsilon e_l M_1OP_1\left\{\left(q\cos\delta - p\sin\delta\right)\left[r\left(\frac{M_2}{M}l_{2y} + \frac{M_1}{M}\left[l_{1x}\sin\delta + l_{1y}\cos\delta\right]\right)\cos\delta - \right.\right.$$
$$\left. -r\left(\frac{M_2}{M}l_{2x} + \frac{M_1}{M}\left[l_{1x}\cos\delta - l_{1y}\sin\delta\right]\right)\sin\delta - \left(r+\sigma\right)l_{1y}\right] +$$
$$+\left(q\sin\delta + p\cos\delta\right)\left[r\left(\frac{M_2}{M}l_{2y} + \frac{M_1}{M}\left[l_{1x}\sin\delta + l_{1y}\cos\delta\right]\right)\sin\delta + \right.$$
$$\left.\left. +r\left(\frac{M_2}{M}l_{2x} + \frac{M_1}{M}\left[l_{1x}\cos\delta - l_{1y}\sin\delta\right]\right)\cos\delta - \left(r+\sigma\right)l_{1x}\right]\right\} + O\left(\varepsilon^2\right)$$

Taking into account that $\left(M_2OP_2 - M_1OP_1\right) = 0$ we can simplify the expression:

$$
\begin{aligned}
T = {} & \frac{1}{2}\left(Ap^2 + Bq^2 + \bar{C}_2 r^2 + \bar{C}_1\left(r+\sigma\right)^2\right) - \\
& -\varepsilon e_B \frac{\bar{A}_1}{2}\left(-p\sin\delta + q\cos\delta\right)^2 + \\
& +\varepsilon e_I \left\{\left[\bar{\beta}_2(\bar{A}_2 - \bar{C}_2)rp + \bar{\alpha}_2(\bar{C}_2 - \bar{B}_2)rq\right] + \left(\bar{A}_1 - \bar{C}_1\right)\left(r+\sigma\right)\left(\bar{\beta}_1\left[p\cos\delta + q\sin\delta\right] - \bar{\alpha}_1\left[-p\sin\delta + q\cos\delta\right]\right)\right\} + \\
& +\varepsilon e_l \left\{ M_2 OP_2 \left[\left(l_{2y} - \frac{M_1}{M}\left[l_{1x}\sin\delta + l_{1y}\cos\delta\right]\right)qr + \left(l_{2x} - \frac{M_1}{M}\left[l_{1x}\cos\delta - l_{1y}\sin\delta\right]\right)pr\right] + \right. \\
& \quad + M_1 OP_1 \left[p\left(\frac{M_1}{M}\left[l_{1x}\cos\delta - l_{1y}\sin\delta\right]r + \left(r+\sigma\right)\left[l_{1y}\sin\delta - l_{1x}\cos\delta\right]\right) + \right. \\
& \qquad \left.\left. + q\left(\frac{M_1}{M}\left[l_{1x}\sin\delta + l_{1y}\cos\delta\right]r - \left(r+\sigma\right)\left[l_{1y}\cos\delta + l_{1x}\sin\delta\right]\right)\right]\right\} + O\left(\varepsilon^2\right)
\end{aligned}
$$

So, the kinetic energy can be divided on “unperturbed” (T_0) and “perturbed” (T_ε) parts:

$$
(1.34)\quad
\begin{cases}
T = T_0 + \varepsilon T_\varepsilon; \\
T_\varepsilon = T_{e_B} + T_{e_I} + T_{e_l}; \\
T_0 = \dfrac{1}{2}\left(Ap^2 + Bq^2 + \bar{C}_2 r^2 + \bar{C}_1\left(r+\sigma\right)^2\right); \\
T_{e_B} = -\dfrac{1}{2} e_B \bar{A}_1\left(-p\sin\delta + q\cos\delta\right)^2; \\
T_{e_I} = e_I\left\{\left[\bar{\beta}_2(\bar{A}_2 - \bar{C}_2)rp + \bar{\alpha}_2(\bar{C}_2 - \bar{B}_2)rq\right] + \right. \\
\qquad \left. + \left(\bar{A}_1 - \bar{C}_1\right)\left(r+\sigma\right)\left(\bar{\beta}_1\left[p\cos\delta + q\sin\delta\right] - \bar{\alpha}_1\left[-p\sin\delta + q\cos\delta\right]\right)\right\}; \\
T_{e_l} = e_l\left\{ M_2 OP_2\left[\left(l_{2y} - \dfrac{M_1}{M}\left[l_{1x}\sin\delta + l_{1y}\cos\delta\right]\right)qr + \left(l_{2x} - \dfrac{M_1}{M}\left[l_{1x}\cos\delta - l_{1y}\sin\delta\right]\right)pr\right] + \right. \\
\qquad + M_1 OP_1\left[p\left(\dfrac{M_1}{M}\left[l_{1x}\cos\delta - l_{1y}\sin\delta\right]r + \left(r+\sigma\right)\left[l_{1y}\sin\delta - l_{1x}\cos\delta\right]\right) + \right. \\
\qquad\qquad \left.\left. + q\left(\dfrac{M_1}{M}\left[l_{1x}\sin\delta + l_{1y}\cos\delta\right]r - \left(r+\sigma\right)\left[l_{1y}\cos\delta + l_{1x}\sin\delta\right]\right)\right]\right\};
\end{cases}
$$

Now it is possible to write the kinetic energy and the Hamiltonian of the coaxial system in the Serret-Andoyer variables in the first order of the approximation (relatively ε) basing on the “conjunctional” expressions (1.31) for the angular momentum components:

$$
(1.35)\quad T\left(l,L,\delta,\Delta,G\right) = T\left(\hat{p},\hat{q},\hat{r},\hat{\sigma}\right) \overset{\varepsilon^2\to 0}{=} Lin\left\{T_0\left(\hat{p},\hat{q},\hat{r},\hat{\sigma}\right)\right\} + \varepsilon T_\varepsilon\left(\hat{p}_0,\hat{q}_0,\hat{r}_0,\hat{\sigma}_0\right);
$$

where

$$
\begin{aligned}
& Lin\left\{T_0\left(\hat{p},\hat{q},\hat{r},\hat{\sigma}\right)\right\} = \frac{1}{2}\left(A\left[\hat{p}_0^2 + 2\varepsilon\hat{p}_0\hat{p}_1\right] + B\left[\hat{q}_0^2 + 2\varepsilon\hat{q}_0\hat{q}_1\right] + \bar{C}_2\left[\hat{r}_0^2 + 2\varepsilon\hat{r}_0\hat{r}_1\right] + \right. \\
(1.36)\quad & \left. + \bar{C}_1\left[\left(\hat{r}_0^2 + 2\varepsilon\hat{r}_0\hat{r}_1\right) + 2\hat{r}_0\hat{\sigma}_0 + 2\varepsilon\left(\hat{r}_0\hat{\sigma}_1 + \hat{r}_1\hat{\sigma}_0\right) + \left(\hat{\sigma}_0^2 + 2\varepsilon\hat{\sigma}_0\hat{\sigma}_1\right)\right]\right) = \\
& = \hat{T}_0\left(\hat{p}_0,\hat{q}_0,\hat{r}_0,\hat{\sigma}_0\right) + \varepsilon\tilde{T}\left(\hat{p}_0,\hat{p}_1,\hat{q}_0,\hat{q}_1,\hat{r}_0,\hat{r}_1,\hat{\sigma}_0,\hat{\sigma}_1\right);
\end{aligned}
$$

and in its turn

(1.37) $$\widehat{T}_0 = \frac{1}{2}\left(A\widehat{p}_0^2 + B\widehat{q}_0^2 + \bar{C}_2\widehat{r}_0^{\,2} + \bar{C}_1\left(\widehat{r}_0 + \widehat{\sigma}_0\right)^2\right); \qquad \tilde{T} = A\widehat{p}_0\widehat{p}_1 + B\widehat{q}_0\widehat{q}_1 + \bar{C}_2\widehat{r}_0\widehat{r}_1 + \bar{C}_1\left[\widehat{r}_0\widehat{r}_1 + \widehat{r}_0\widehat{\sigma}_1 + \widehat{r}_1\widehat{\sigma}_0 + \widehat{\sigma}_0\widehat{\sigma}_1\right]$$

So, we have the final expression for the kinetic energy written in the Serret-Andoyer variables

(1.38) $$T(l,L,\delta,\Delta,G) = \widehat{T}_0\left(\widehat{p}_0,\widehat{q}_0,\widehat{r}_0,\widehat{\sigma}_0\right) + \varepsilon\widehat{T}_1\left(\widehat{p}_0,\widehat{p}_1,\widehat{q}_0,\widehat{q}_1,\widehat{r}_0,\widehat{r}_1,\widehat{\sigma}_0,\widehat{\sigma}_1\right);$$

$$\widehat{T}_1\left(\widehat{p}_0,\widehat{p}_1,\widehat{q}_0,\widehat{q}_1,\widehat{r}_0,\widehat{r}_1,\widehat{\sigma}_0,\widehat{\sigma}_1\right) = \tilde{T}\left(\widehat{p}_0,\widehat{p}_1,\widehat{q}_0,\widehat{q}_1,\widehat{r}_0,\widehat{r}_1,\widehat{\sigma}_0,\widehat{\sigma}_1\right) + T_\varepsilon\left(\widehat{p}_0,\widehat{q}_0,\widehat{r}_0,\widehat{\sigma}_0\right).$$

If we reduce the corresponding perturbed parts/terms, then the expression follows:

(1.39) $$\widehat{T}_1 = T_{\mathbf{I}} + T_B + T_l$$

where

(1.40) $$T_B = \frac{1}{2}e_B\bar{A}_1\left(G^2 - L^2\right)\left(\frac{1}{A}\sin l\sin\delta - \frac{1}{B}\cos l\cos\delta\right)^2;$$

(1.41) $$T_{\mathbf{I}} = -e_{\mathbf{I}}\sqrt{G^2 - L^2}\left\{\frac{L-\Delta}{\bar{C}_2}\left(\bar{\beta}_2\left(\bar{A}_2 - \bar{C}_2\right)\frac{\sin l}{A} + \bar{\alpha}_2\left(\bar{C}_2 - \bar{B}_2\right)\frac{\cos l}{B}\right) + \right.$$

$$\left. + \left(\bar{A}_1 - \bar{C}_1\right)\frac{\Delta}{\bar{C}_1}\left[\frac{\sin l}{A}\left(\bar{\beta}_1\cos\delta + \bar{\alpha}_1\sin\delta\right) + \frac{\cos l}{B}\left(\bar{\beta}_1\sin\delta - \bar{\alpha}_1\cos\delta\right)\right]\right\};$$

(1.42) $$T_l = e_l\sqrt{G^2 - L^2}\left\{L\left[\frac{M_1\cos l}{B\bar{C}_2 M}\left(\left[l_{1x}\sin\delta + l_{1y}\cos\delta\right]M_1 OP_1 - l_{2y}M_2 OP_2\right) + \right.\right.$$

$$\left. + \frac{M_1\sin l}{A\bar{C}_2 M}\left(\left[l_{1x}\cos\delta - l_{1y}\sin\delta\right]M_1 OP_1 - l_{2x}M_2 OP_2\right)\right] +$$

$$\left. + \frac{\Delta\left(OP_1\bar{C}_2 - OP_2\bar{C}_1\right)M_1M_2}{\bar{C}_1\bar{C}_2 M}\left[\frac{\sin l}{A}\left(l_{1x}\cos\delta - l_{1y}\sin\delta - l_{2x}\right) + \frac{\cos l}{B}\left(l_{1x}\sin\delta + l_{1y}\cos\delta - l_{2y}\right)\right]\right\};$$

1.3.3. External and internal perturbing torques

In addition to the described "asymmetry perturbations", the natural/artificial external/internal disturbing forces/torques can be presented. These forces/torques are possible due to actions of external gravitational and/or electromagnetic fields, due to the existence of dissipations from the rarefied atmosphere and/or the internal friction of the system's bodies, due to implementations of the control systems signals by actuators, etc.

1.3.3.1. The magnetic torques

As the basic case we will consider the angular/attitude motion of the relatively small DSSC with a permanent magnet (placed along the longitudinal axis of the DSSC) about its center of mass at the implementation of the orbital motion along the equatorial circle orbit of the Earth.

In this case, as it was shown in the works [Doroshin (2013b), (2015)], we can, firstly, neglect the influence of the gravitational torque and, secondly, consider the vector of the Earth magnetic field's induction $\mathbf{B}_{\mathbf{orb}}$ as the constant vector (in the inertial space *CXYZ*) with corresponding magnet torques initiation like in the well-known Lagrange-top.

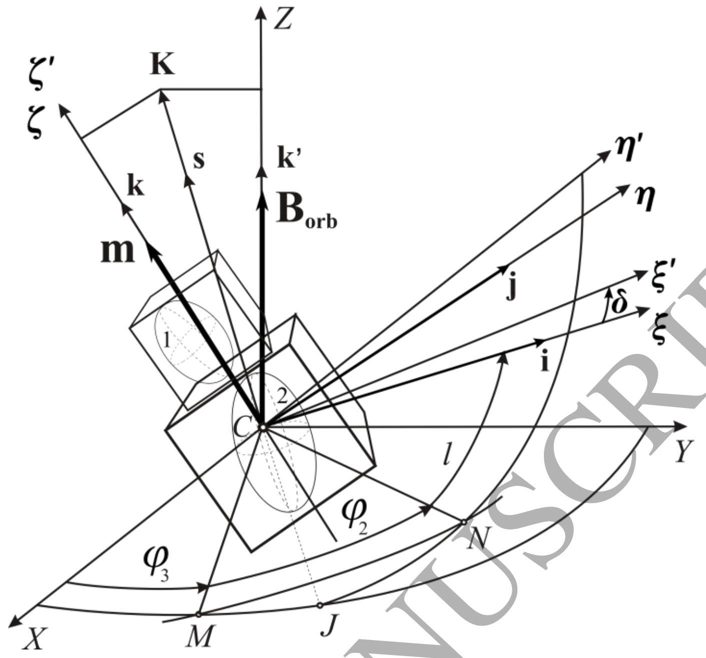

Fig.2 The Serret-Andoyer variables

Let us assume that the DSSC own magnetic dipole moment **m** is created by some electromagnetic equipment (permanent magnets in the simplest case) placed in the rotor-body. We will consider the case when this dipole moment has a preferably longitudinal (along P_1z') direction (with the corresponding component *m*) and small (~ε) constant components along the equatorial rotor-axis (P_1x', P_1y'):

$$(1.43)\quad \begin{cases} Ox_0'y_0'z_0': \ \mathbf{m}' = \left[m_{x'},\ m_{y'},\ m\right]^T; \\ m_{x'} = m_\perp \cos\chi; \ \ m_{y'} = m_\perp \sin\chi; \ \ m_\perp = \sqrt{m_{x'}^2 + m_{y'}^2} = \text{const}; \\ \chi = \text{const}; \ \ m_\perp \ll m; \ \ m_\perp = \varepsilon e_m m \end{cases}$$

where the angular parameter χ is artificially introduced for the convenience of the interpretation of the magnet dipole's "equatorial" projections disposition. The multiplier $e_m = (m_\perp / m)/\varepsilon$ can be interpreted as the dimensionless scale factor relative to the main small parameter ε.

So, in the indicated case (the corresponding conditions and features are described in works [Doroshin (2013b), (2015)]) we can consider the attitude motion of the DSSC as the motion of the coaxial rigid-bodies-system around the fixed point under the action of restoring/overturning torques which are analogous to the generalized coaxial Lagrange-top.

The corresponding magnet torque has the form

$$(1.44)\quad \mathbf{M}_{\mathbf{m}} = \mathbf{m} \times \mathbf{B}_{\mathbf{orb}}$$

which can be written in the projections onto axes $C\xi\eta\varsigma$:

(1.45) $\mathbf{M_m} = mB_{orb}\left([\boldsymbol{\delta}']\cdot[\varepsilon e_m \cos\chi,\ \ \varepsilon e_m \cos\chi,\ \ 1]^T\right)\times[\gamma_1,\gamma_2,\gamma_3]^T =$

$$= mB_{orb}\left(\begin{bmatrix}-\gamma_2\\ \gamma_1\\ 0\end{bmatrix} + \varepsilon e_m \begin{bmatrix}\gamma_3\left(c_\chi \sin\delta + s_\chi \cos\delta\right)\\ -\gamma_3\left(c_\chi \cos\delta - s_\chi \sin\delta\right)\\ \gamma_2\left(c_\chi \cos\delta - s_\chi \sin\delta\right) - \gamma_1\left(c_\chi \sin\delta + s_\chi \cos\delta\right)\end{bmatrix}\right);$$

where $s_\chi = \sin\chi$; $c_\chi = \cos\chi$; the vector $[\gamma_1,\gamma_2,\gamma_3]^T$ defines the "vertical" inertial direction CZ coinciding with the constant position of the vector $\mathbf{B_{orb}}$, and also it corresponds to the vector of directional cosines of this "vertical" axes in the frame $C\xi\eta\varsigma$ which has the following components depending on the Euler angles (fig.3):

(1.46) $\gamma_1 = \sin\theta\sin\varphi;\ \ \gamma_2 = \sin\theta\cos\varphi;\ \ \gamma_3 = \cos\theta$

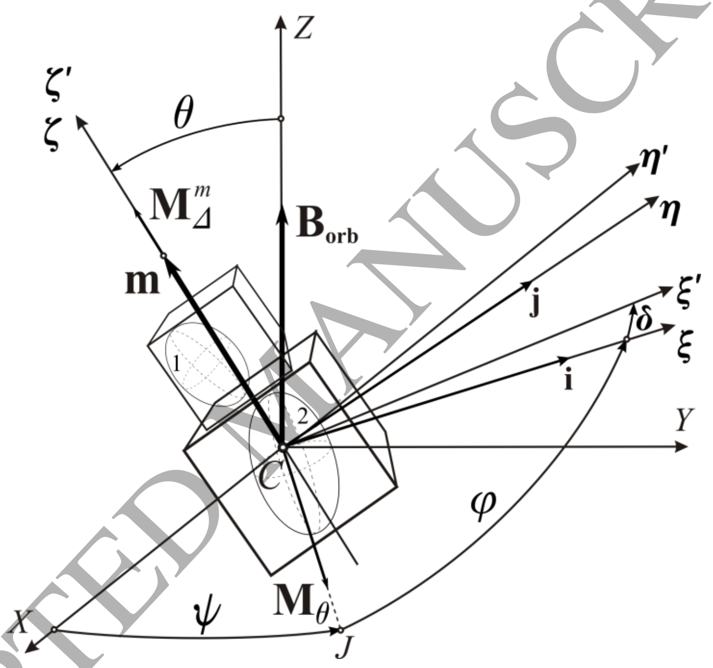

Fig. 3. The Euler angles

It is needed to note that the magnetic torque can be decomposed on two torques: the restoring/overturning torque $\mathbf{M}_\theta$ and the small rotational torque $\mathbf{M}_\Delta^m$. This restoring/overturning torque $\mathbf{M}_\theta$ corresponds to the Euler-torque along the "nodes line" CJ, that is formed by the projection of the magnetic torque $\mathbf{M_m}$ on the "nodes line"; and the small rotational torque is the projection of the magnetic moment on the longitudinal axis $C\varsigma$:

$$(1.47)\quad \begin{cases} M_\theta = M_m^\xi \cos\varphi - M_m^\eta \sin\varphi = \\ \quad = mB_{orb}\left(-\sin\theta + \varepsilon e_m \cos\theta\left[\left(c_\chi \sin\delta + s_\chi \cos\delta\right)\cos\varphi + \left(c_\chi \cos\delta - s_\chi \sin\delta\right)\sin\varphi\right]\right); \\ M_\Delta^m = M_m^\zeta = \varepsilon e_m m B_{orb} \sin\theta\left(\left(c_\chi \cos\delta - s_\chi \sin\delta\right)\cos\varphi - \left(c_\chi \sin\delta + s_\chi \cos\delta\right)\sin\varphi\right) \end{cases}$$

Now we can write these torques as the generalized forces defined by the potential energy (or, that is the same, we can define the corresponding potential energy):

(1.48) $P = -\int M_\theta d\theta = P_0 + \varepsilon P_1; \qquad P_0 = -mB_{orb}\cos\theta;$

$$P_1 = -e_m m B_{orb} \sin\theta \left[\left(c_\chi \sin\delta + s_\chi \cos\delta \right) \cos\varphi + \left(c_\chi \cos\delta - s_\chi \sin\delta \right) \sin\varphi \right]$$

It is possible to make sure that the correspondences are correct:

(1.49) $M_\theta = -\dfrac{\partial P}{\partial \theta}; \qquad M_\Delta^m = -\dfrac{\partial P}{\partial \delta} = -\varepsilon \dfrac{\partial P_1}{\partial \delta}$

So, we further will consider the important unperturbed ($\varepsilon = 0$) case of the motion, when the vector of the initial angular momentum of the system is coincided with the "immovable" vector $\mathbf{B_{orb}}$: $\mathbf{K} \uparrow\uparrow \mathbf{B_{orb}} \uparrow\uparrow CZ$; then the system angular momentum also will be the constant vector which is motionless in the inertial space $|\mathbf{K}| = K = K_Z$ since in unperturbed cases $\mathbf{M}_\Delta^m = 0$, $\mathbf{M}_\theta \perp CZ$ (it does not change the "vertical" *CZ*-component of the angular momentum), and, therefore, $|\mathbf{K}| = \mathrm{const}$. Then, turning to the Serret-Andoyer variables (1.30), we can write the following important expressions, which connect the Serret-Andoyer variables with the Euler angles:

(1.50) $\begin{cases} H = G = \mathrm{const}; \ \ \cos\theta = L/H = L/G; \quad \sin\theta = \sqrt{G^2 - L^2}/G; \\ \varphi_3 \equiv 0; \ \ \varphi_2 = \psi; \ \ l = \varphi \end{cases}$

Therefore, as it was also indicated in the previous works [Doroshin (2013a), (2013b)], the Serret-Andoyer angle l in the considering case is fully corresponds to the intrinsic rotation angle (l=φ); and the nutation angle θ cosine is corresponds to the relation of the Serret-Andoyer impulses, and the angle φ_2 is, per se, the precession-angle.

So, with the help of formulas (1.50) the potential energy (fully describing the magnetic torque (1.44)) can be written in the Serret-Andoyer canonical variables:

(1.51) $P = P_0 + \varepsilon P_1^m; \qquad P_0 = Q\dfrac{L}{G};$

$$P_1^m = e_m Q \frac{\sqrt{G^2 - L^2}}{G} \left[\left(c_\chi \sin\delta + s_\chi \cos\delta \right) \cos l + \left(c_\chi \cos\delta - s_\chi \sin\delta \right) \sin l \right];$$

where

$$Q = -mB_{orb} = \mathrm{const}.$$

1.3.3.2. The friction torque between the coaxial bodies

Let us assume the presence of the small friction torque between the coaxial DSSC's bodies (affecting the angle of the relative rotation δ), which has a two-part form, collecting the "liquid-type" friction (M_Δ^{fl}) and the "dry-type" friction (M_Δ^{fd}), depending on the value of the relative angular velocity (σ), and on its sign, respectively:

(1.52) $M_{\Delta}^{fl}=-\nu\sigma;\qquad M_{\Delta}^{fd}=-\operatorname{sign}(\sigma)\kappa=-\kappa\frac{\sigma}{|\sigma|};\qquad M_{\Delta}^{f}=M_{\Delta}^{fl}+M_{\Delta}^{fd}=-\varepsilon\sigma\left[e_{\nu}+e_{\kappa}/|\sigma|\right],$

where $e_{\nu}=\nu/\varepsilon$, $e_{\kappa}=\kappa/\varepsilon$ are the dimensional scale factors of the small friction's parts. Taking into account only terms of the first smallness order (~ε) and basing on the expressions (1.31), (1.32), (1.33) we obtain the form of the friction torque in the Serret-Andoyer canonical variables:

(1.53) $M_{\Delta}^{f}=\varepsilon m_{\Delta}^{f};\qquad m_{\Delta}^{f}=-\frac{1}{\bar{C}_1\bar{C}_2}\left[\left(\bar{C}_1+\bar{C}_2\right)\Delta-\bar{C}_1L\right]\left[e_{\nu}+e_{\kappa}\bar{C}_1\bar{C}_2/\left|\left(\bar{C}_1+\bar{C}_2\right)\Delta-\bar{C}_1L\right|\right],$

or in the form

(1.54) $$\begin{cases} m_{\Delta}^{f}=-\frac{e_{\nu}}{\bar{C}_1\bar{C}_2}\left[\left(\bar{C}_1+\bar{C}_2\right)\Delta-\bar{C}_1L\right]+e_{\kappa}F(t)=-e_{\nu}\sigma(t)+e_{\kappa}F(t), \\ F(t)=-\operatorname{sign}\left(\frac{e_{\nu}}{\bar{C}_1\bar{C}_2}\left[\left(\bar{C}_1+\bar{C}_2\right)\Delta-\bar{C}_1L\right]\right)=-\operatorname{sign}\left(\sigma(t)\right) \end{cases}$$

1.3.3.3. The electromagnetic internal torque of the spinning up the rotor

Let us define the form of the torque into the internal DSSC-rotor-electromotor (DC), which we can use for the modeling spin-up dynamics and stabilizing of the constant angular velocity of the rotor-body of the DSSC. We can simulate the magnitude of the electromotor's torque as proportional to the current intensity (*I*); and in turn the current intensity depends on the voltage (*U*), on the counterelectromotive force (E_B) and on the electric resistance (*R*) of the circuit:

(1.55) $M_{\Delta}^{DC}=k_I I;\qquad E_B=k_B\sigma;\qquad U=IR+E_B$

From the equations (1.55) the DC-motor's torque expression follows

$$M_{\Delta}^{DC}(t)=\frac{k_I}{R}\left(U(t)-k_B\sigma\right)=k_U U(t)-\mu\sigma;$$
$$k_U=k_I/R=\text{const};\quad \mu=k_I k_B/R=\text{const}$$

So, we, as in the past sections, will consider the case of perturbations, when the torques are small:

(1.56) $M_{\Delta}^{DC}=\varepsilon m_{\Delta}^{DC};\; m_{\Delta}^{DC}=e_U U(t)-e_{\mu}\frac{1}{\bar{C}_1\bar{C}_2}\left[\left(\bar{C}_1+\bar{C}_2\right)\Delta-\bar{C}_1L\right];\quad e_U=k_U/\varepsilon;\quad e_{\mu}=\mu/\varepsilon,$

or in the form

(1.57) $$\begin{cases} m_{\Delta}^{DC}=-e_{\mu}\sigma(t)+e_U F(t), \\ F(t)=U(t) \end{cases}$$

1.3.3.4. The positional polyharmonic torque from the control system

Additionally the control torque can act on the coaxial rotor from the side of the internal spinup-engine which is controlled by the control/stabilizing system.

This torque can correspond to the special control signal for tracking of the complex time-dependence; or it can be piecewise constant (with alternating/constant signs) in order to create corresponding piecewise spin-up/spin-down maneuvers of the rotor; or it presents a trigger mode to maintain a nominal value of the relative angular velocity, etc. Also unwanted spurious control signals are possible (e.g. small harmonic signals in stabilization systems at delays in feedback loops) which lead to corresponding spurious torques.

Let us consider the case of the polyharmonic spurious torque which is actual practically in any case of periodical perturbations and corresponds to the general form of the expansion in Fourier series by the relative rotation δ phase, taking into account N harmonic components [Doroshin (2014c), (2015)]:

(1.58) $$M_{\Delta}^{\delta} = \varepsilon m_{\Delta}^{\delta}; \qquad m_{\Delta}^{\delta} = e_{\delta} \sum_{n=1}^{N} \left[\bar{a}_n \sin(n\delta) + \bar{b}_n \cos(n\delta) \right]$$

where $\bar{a}_n$, $\bar{b}_n$ are the constant Fourier coefficients, and e_{δ} is the scale factor of the polyharmonic perturbation smallness.

Formally we can present the torque (1.58) as the potential one with the corresponding small part of the potential energy:

(1.59) $$P_1^{\delta} = -\int m_{\Delta}^{\delta} d\delta = e_{\delta} \sum_{n=1}^{N} \left[a_n \sin(n\delta) + b_n \cos(n\delta) \right]; \qquad a_n = -\frac{\bar{b}_n}{n}; \qquad b_n = \frac{\bar{a}_n}{n}$$

Here we note that this polyharmonic form can be considered as the general Fourier series expansion of the arbitrary control torques acting in the internal DSSC rotor-engine.

1.3.4. The perturbed Hamiltonian and the perturbed equations system

The constructed above expressions for the kinetic energy (1.38) (with parts (1.39), (1.40), (1.41), (1.42)) and potential energy terms (1.51), (1.59) at the conditions (1.50) allow us to write the following Hamiltonian ($\mathcal{H}$):

(1.60) $$\mathcal{H} = \mathcal{H}_0 + \varepsilon \mathcal{H}_1; \quad \mathcal{H}_0 = T_0 + P_0; \qquad \mathcal{H}_1 = T_1 + P_1;$$

where

(1.61) $$\begin{cases} T_0 = \dfrac{G^2 - L^2}{2} \left[\dfrac{1}{A} \sin^2 l + \dfrac{1}{B} \cos^2 l \right] + \dfrac{1}{2} \left[\dfrac{\Delta^2}{\bar{C}_1} + \dfrac{(L - \Delta)^2}{\bar{C}_2} \right]; \qquad P_0 = Q \dfrac{L}{G}; \\ \qquad T_1 = T_{\mathbf{I}} + T_B + T_l; \qquad P_1 = P_1^m + P_1^{\delta} \end{cases}$$

The corresponding dynamical equations in the canonical Serret-Andoyer variables are

(1.62) $$\dot{L} = f_L + \varepsilon g_L; \quad \dot{\Delta} = f_{\Delta} + \varepsilon g_{\Delta}; \quad \dot{l} = f_l + \varepsilon g_l; \qquad \dot{\delta} = f_{\delta} + \varepsilon g_{\delta};$$

where the following right-parts-functions take place (including the perturbations (1.53), (1.56))

(1.63) $$\begin{cases} f_L = -\dfrac{\partial \mathcal{H}_0}{\partial l}; & f_l = \dfrac{\partial \mathcal{H}_0}{\partial L}; & f_\Delta = -\dfrac{\partial \mathcal{H}_0}{\partial \delta} = 0; & f_\delta = \dfrac{\partial \mathcal{H}_0}{\partial \Delta}; \\ g_L = -\dfrac{\partial \mathcal{H}_1}{\partial l}; & g_l = \dfrac{\partial \mathcal{H}_1}{\partial L}; & g_\Delta = -\dfrac{\partial \mathcal{H}_1}{\partial \delta} + m_\Delta^f + m_\Delta^{DC}; & g_\delta = \dfrac{\partial \mathcal{H}_1}{\partial \Delta}; \end{cases}$$

so, we can note the presence of the non-Hamiltonian parts $\left(m_\Delta^f,\ m_\Delta^{DC}\right)$ in the dynamical equation for Δ, which disrupts the system conservativeness. Also it should be underlined that the equations (1.62) correspond to the fourth-order-system, but the considering DSSC mechanical system has four degrees-of-freedom and, therefore, we must formally add four equations for unnoticed canonical coordinates and momentums:

(1.64) $$\dot{G} = -\frac{\partial \mathcal{H}}{\partial \varphi_2} = 0; \quad \dot{\varphi}_2 = \frac{\partial \mathcal{H}}{\partial G}; \quad \dot{H} = -\frac{\partial \mathcal{H}}{\partial \varphi_3} = 0; \quad \dot{\varphi}_3 = \frac{\partial \mathcal{H}}{\partial H} = 0$$

The equations (1.64) present the subsystem of the complete dynamical model for the cyclic coordinates, which do not affect the dynamics of the positional variables, and we can consider the system (1.62) as the main independent dynamical system.

2. The heteroclinic solutions for the attitude dynamics of the DSSC

It is well-known fact that the separatrix regions of the dynamical systems phase portraits are very sensitive to perturbations, that results in a homo(hetero)clinic net generation, in a dynamical intricacy, and in the dynamical chaos appearance. In the purposes of the investigation ща ерy possible local dynamical chaotization (including tasks of the chaotic regimes avoidance) we need, first of all, to have exact analytical time-dependencies for the corresponding unperturbed homo(hetero)clinic separatrix. Based on the works of H.Poincaré, V.I.Arnold, V.K.Melnikov, P.J.Holmes, J.E.Marsden, J.Guckenheimer, S.Wiggins, V.V.Kozlov, A.I.Neishtadt and other well-known researchers, these dependencies are used for the homo(hetero)clinic tangles detection and for the parametrical synthesis for the chaotic regimes avoidance.

2.1. The main heteroclinic solutions for the angular velocity components

So, as it was indicated above, generating unperturbed exact solutions/dependencies for the homo(hetero)clinic separatrix are necessary for the local dynamical chaos analysis.

As these generating solutions in this research we will use the heteroclinic solutions for magnetic DSSC which moved along the equatorial circle Earth orbit [Doroshin (2012), (2013b), (2015)] at the condition of the “cylindrical precession” realization. It means that the vector of the system initial angular momentum is coincided with the “immovable” (in the framework of the considered research formulation) vector $\mathbf{B}_{\mathbf{orb}}$, so the expressions (1.50) are fulfilled.

These solutions generalize corresponding solutions for the free DSSC motion, and can be applied to the chaotic DSSC dynamics investigation in the considering case.

The following generating (ε=0) heteroclinic solutions $\{\bar{p}(t), \bar{q}(t), \bar{r}(t), \bar{\sigma}(t)\}$ are actual [Doroshin (2015)]:

$$(2.1)\quad \begin{cases} \bar{p}(t) = \pm\sqrt{\dfrac{\bar{C}_2\left(B-\bar{C}_2\right)}{A(A-B)}}\,y(t); \quad \bar{q}(t) = \pm\sqrt{s^2 - k^2\left(y(t)+\Delta\beta+E\alpha\right)^2}; \\ \bar{r}(t) = y(t) + \dfrac{\Delta - EB}{B-\bar{C}_2}; \quad \bar{\sigma}(t) = \dfrac{\Delta}{\bar{C}_1} - \bar{r}(t) = \left[\dfrac{\Delta}{\bar{C}_1} - \dfrac{\Delta - EB}{B-\bar{C}_2}\right] - y(t); \end{cases}$$

where

$$(2.2)\quad y(t) = \frac{4a_0\hat{E}_a\exp(\lambda t)}{\left[\hat{E}_a\exp(\lambda t) - a_1\right]^2 - 4a_2a_0},$$

with the following set of the values:

$$\Delta = \text{const} > 0;\ a_2 = -k^2 < 0;\ a_1 = -2(\Delta\beta + E\alpha)k^2;\ a_0 = s^2 - k^2(\Delta\beta+E\alpha)^2;\ k^2 = \frac{\bar{C}_2\left(A-\bar{C}_2\right)}{B(A-B)};$$

$$y_0^{\pm} = \pm\frac{s}{k} - (\Delta\beta + E\alpha);\ \alpha = \frac{A}{A-\bar{C}_2} - \frac{B}{B-\bar{C}_2};\ \beta = \frac{1}{B-\bar{C}_2} - \frac{1}{A-\bar{C}_2};\ s^2 = \frac{\tilde{H}}{B(A-B)};$$

$$\tilde{H} = 2\tilde{T}\left(A-\tilde{D}\right) + \frac{\bar{C}_2}{A-\bar{C}_2}(\Delta - EA)^2 - \left[\left(\frac{A}{\bar{C}_1} - 1\right)\Delta^2 + 2EA\Delta\right];\ \tilde{T} = T_0 - \tilde{Q}\frac{\bar{C}_2 r_0 + \Delta}{K};$$

$$2T_0 = Ap_0^2 + Bq_0^2 + \bar{C}_2 r_0^2 + \frac{\Delta^2}{\bar{C}_1};\quad K^2 = A^2p_0^2 + B^2q_0^2 + \left[\bar{C}_2 r_0 + \Delta\right]^2;\quad \tilde{Q} = -Q = \pm\left|mB_{orb}\right|;$$

$$E_a(z) \overset{df}{=} \left[\frac{2a_0 + a_1 z + 2\sqrt{a_0}\sqrt{a_2 z^2 + a_1 z + a_0}}{z}\right];\quad \hat{E}_a = E_a\left(y_0^{\pm}\right);\quad \lambda = \mp\frac{\tilde{M}\sqrt{a_0}}{k^2};$$

$$\tilde{D} = B + \frac{1}{2\tilde{T}}\left(\frac{\bar{C}_2}{B-\bar{C}_2}(\Delta - EB)^2 - \left[\left(\frac{B}{\bar{C}_1} - 1\right)\Delta^2 + 2EB\Delta\right]\right);\ E = -\frac{\tilde{Q}}{K};\ \tilde{M} = \frac{\left(A-\bar{C}_2\right)}{B}\sqrt{\frac{\bar{C}_2\left(B-\bar{C}_2\right)}{A(A-B)}}$$

As it follows from the last solution (2.1), the heteroclinic solution for δ takes place:

$$\bar{\delta} = \int\bar{\sigma}(t)dt = \int\left[\frac{\Delta}{\bar{C}_1} - \bar{r}(t)\right]dt = \left[\frac{\Delta}{\bar{C}_1} - \frac{\Delta - EB}{B-\bar{C}_2}\right]t - \int y(t)\,dt;$$

or in the reduced shape:

$$(2.3)\quad \bar{\delta} = \sigma_* t - \bar{v}_\delta(t) + \delta_0;$$

where

$$(2.4)\quad \sigma_* = \left[\frac{\Delta}{\bar{C}_1} - \frac{\Delta - EB}{B-\bar{C}_2}\right];$$

$$(2.5)\quad \bar{v}_\delta(t) = \Phi(t) - \Phi(0);$$

$$(2.6)\quad \Phi(t) = \int y(t)\,dt = \frac{2}{\lambda}\sqrt{\frac{a_0}{|a_2|}}\operatorname{arctg}\left[\frac{\hat{E}_a\exp(\lambda t) - a_1}{2\sqrt{a_0|a_2|}}\right].$$

The same form of the quadrature (2.6) was also obtained by the author in the article [Aslanov, Doroshin (2010)], where analogous $\bar{\delta}$ – solution was found for the case of the non-magnetic gyrostat (when the solutions [Doroshin (2012)] are also useful). Here additionally we can indicate that the function $y(t)$ is even function damped to zero at $t\to\pm\infty$ (fig.4); and the function $\bar{v}_\delta(t)$ is the odd-function (fig.4) saturated to the value $\pm v_*$:

$$(2.7)\qquad \Phi(+\infty)=\frac{\pi}{\lambda}\sqrt{\frac{a_0}{|a_2|}}=\pi\sqrt{\frac{AB}{\left(B-\bar{C}_2\right)\left(A-\bar{C}_2\right)}};\qquad v_*=\pi\sqrt{\frac{AB}{\left(B-\bar{C}_2\right)\left(A-\bar{C}_2\right)}}-\Phi(0)$$

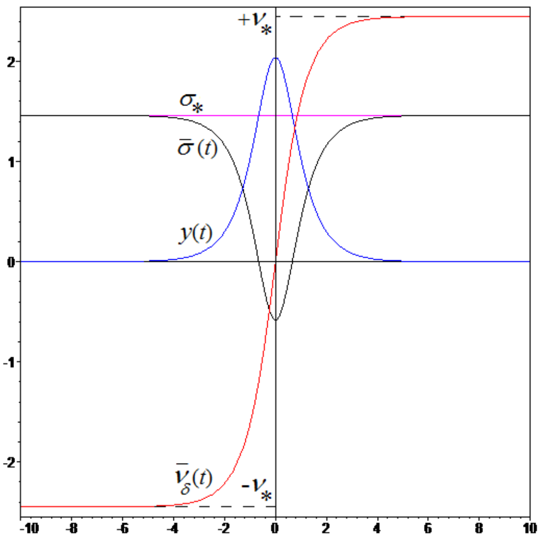

Fig.4. The main heteroclinic time-dependencies

Also it is possible to write the corresponding obvious heteroclinic solutions for the Serret-Andoyer variables in the generating case (ε=0) using the connections (1.32):

$$(2.8)\qquad \bar{L}(t)=\bar{C}_2\bar{r}(t)+\Delta;\quad \sin\left(\bar{l}(t)\right)=\frac{A\bar{p}(t)}{\sqrt{G^2-\bar{L}^2(t)}};\quad \left(\text{or}\quad \cos\left(\bar{l}(t)\right)=\frac{B\bar{q}(t)}{\sqrt{G^2-\bar{L}^2(t)}}\right)$$

The solutions (2.1), (2.6) and (2.8) are used in the next part of this paper for the chaotic aspects analysis.

2.2. The heteroclinic solutions for the action-angle variables of the coaxial rotor relative rotation

In purposes of the S.Wiggins' methodology [Wiggins (1988)] application for detecting intersections of the split separatrix homo(hetero)clinic manifolds, we need to use the form of the dynamical system with the reduction of the rotating-phase-variable to its corresponding action-angle type. We can additionally indicate, the same transformations of the variables to the action-angle types are also needed for the application of other modified Melnikov's-methodologies, including the Arnold's [Arnold (1964)] and the Holms'-Marsden's methodologies [Holmes & Marsden (1983)]. At least, these methodologies require using reducible variables which "play roles" of the action-angle-variables [Holmes & Marsden (1983)].

In our case the δ-variable plays the role of the rotating-phase-variable (with the corresponding canonical momentum Δ), and, therefore, we need to obtain the modified form of the dynamical system (1.62) at the reduction of this canonical pair $\{\delta, \Delta\}$ to the corresponding canonical action-angle pair $\{w_\delta, I_\Delta\}$. It is also important to remind the obtained results for the decomposition of variables in the task of a gyrostat motion [Ivin (1985)], and for action-integrals [Aslanov (2012)].

In the framework of the action-angle pair $\{w_\delta, I_\Delta\}$ construction, we have to note, firstly, that the "generating function" must be constructed for the implementation of the transition between the canonical variables $\{\delta, \Delta\} \to \{w_\delta, I_\Delta\}$. Secondly and most importantly, for using the S.Wiggins methodology we should obtain the explicit exact solution form of this action-angle pair on the heteroclinic separatrix $\{\bar{w}_\delta, \bar{I}_\Delta\}$.

So, we will find the generating function of the canonical transformation in the general form

(2.9) $\quad W = W\left(t, q_i, \tilde{p}_i\right)$

where $\{q_i, \tilde{q}_i\}$ symbolize the "old" and "new" canonical coordinates, and $\{p_i, \tilde{p}_i\}$ are the "old" and "new" canonical impulses. Then the correspondences take place ($\tilde{\mathcal{H}}$ is the "new" Hamiltonian)

(2.10) $\quad \dfrac{\partial W}{\partial q_i} = p_i; \qquad \dfrac{\partial W}{\partial \tilde{p}_i} = \tilde{q}_i; \qquad \tilde{\mathcal{H}} = \mathcal{H} + \dfrac{\partial W}{\partial t}$

The concretized shape of the generating function we can obtain basing on works [Arkhangelskiĭ U.A. (1977)] and [Sadov Y.A. (1970)].

As it possible to see, for our task the first integrals are relevant

(2.11) $\quad \alpha_1 = \text{const} = \mathcal{H}_0\left(l, L, \delta, \Delta\right); \qquad \alpha_2 = \text{const} = \Delta$

Now we formally solve the expressions (2.11) as the equations relative the momentums

(2.12) $\quad L = L\left(l, \alpha_1, \alpha_2\right) = F_L\left(l, \alpha_1, \alpha_2\right); \qquad \Delta = \Delta\left(\delta, \alpha_1, \alpha_2\right) = F_\Delta\left(\delta, \alpha_1, \alpha_2\right)$

where $F_\Delta\left(\delta, \alpha_1, \alpha_2\right) = \Delta = \alpha_2$ and the concretized form of the exact dependence $F_L\left(l, \alpha_1, \alpha_2\right)$ can be easy obtained from the first expression (2.11) as the solution of the quadratic algebraic equation, but we leave it in the general form without changing (it will not be needed further).

The formal expressions for the actions are (here closed integrals are taken over a total period of the integrated function, or in case of its constancy – on the 2π-interval of the arguments' variation):

(2.13) $\quad I_L = \dfrac{1}{2\pi} \oint F_L\left(l, \alpha_1, \alpha_2\right) dl;$

(2.14) $\quad I_\Delta = \dfrac{1}{2\pi} \oint F_\Delta\left(\delta, \alpha_1, \alpha_2\right) d\delta = \Delta = \alpha_2$

From the last equalities (2.14), (2.13) we formally express constants α_2, α_1 through the action-constants I_Δ, I_L:

(2.15) $\quad \alpha_2 = \Delta = I_\Delta;$

(2.16) $\quad \alpha_1 = \alpha_1\left(I_L, I_\Delta\right);$

where we do not find the explicit form of the expression (2.16).

Then the concretized shape of the generating function can be written as the complete integral from the following total differential [Arkhangelskiĭ (1977); Sadov (1970)]:

(2.17) $$W = \int \left(F_L \left(l, \alpha_1, \alpha_2 \right) dl + \Delta d\delta \right) = \int F_L \left(l, \alpha_1, \alpha_2 \right) dl + \Delta\delta,$$

where we formally replace the constants α_2, α_1 through the action-constants I_Δ, I_L using (2.15) and (2.16):

(2.18) $$W = \int \left(F_L \left(l, \alpha_1 \left(I_L, I_\Delta \right), I_\Delta \right) dl + I_\Delta d\delta \right) = \int F_L \left(l, \alpha_1 \left(I_L, I_\Delta \right), I_\Delta \right) dl + I_\Delta \delta$$

Now taking into account (2.10) we obtain the formal form of the rotating-phase-angle

(2.19) $$w_\delta = \frac{\partial W}{\partial I_\Delta} = \delta + \frac{\partial}{\partial I_\Delta} \int F_L \left(l, \alpha_1 \left(I_L, I_\Delta \right), I_\Delta \right) dl$$

The reverse interpretation of the formula (2.19) is possible: the Cartesian angular coordinate δ is expressed through the “new” rotating-phase-angle w_δ and the “old” Serret-Andoyer angular coordinate l (also the actions-constants are included) by the following manner:

(2.20) $$\delta = \delta \left(w_\delta, l \right) = w_\delta - v_\delta \left(l, I_L, I_\Delta \right);$$

where

(2.21) $$v_\delta \left(l, I_L, I_\Delta \right) = \frac{\partial}{\partial I_\Delta} \int F_L \left(l, \alpha_1 \left(I_L, I_\Delta \right), I_\Delta \right) dl$$

Also from the expression (2.20) the partial differentiation’s equivalence follows:

(2.22) $$\frac{\partial(\bullet)}{\partial w_\delta} = \frac{\partial(\bullet)}{\partial \delta} \cdot \frac{\partial \delta}{\partial w_\delta} = \frac{\partial(\bullet)}{\partial \delta}$$

If we remind that the canonical variable w_δ belongs to the action-angle-type of variables, then, based on their main properties, the “action” (I_Δ) is constant, and the “angle” is a linear time-function ($w_\delta = \omega_\delta \cdot t + w_0$). In our research we will focus on the heteroclinic orbits, and therefore it is important to obtain the heteroclinic solution for the rotating-phase-angle w_δ. As can we see from (2.3), the heteroclinic solution for the Cartesian angle δ contains the main linear time-part $\sigma_* t$ and the additional aperiodic saturable term $\bar{v}_\delta(t)$. If we compare the structure of the $\bar{\delta}-$heteroclinic solution (2.3) with the structure involved by (2.20), then we explicitly conclude that the equalities are fulfilled

(2.23) $$\begin{aligned} \bar{v}_\delta(t) &= v_\delta \left(\bar{l}(t), \bar{I}_L, \bar{I}_\Delta \right) \Big|_{\{\bar{p},\bar{q},\bar{r},\bar{\sigma},\bar{\delta}\}} = \\ &= \left[\frac{\partial}{\partial I_\Delta} \int F_L \left(\bar{l}, \alpha_1 \left(\bar{I}_L, \bar{I}_\Delta \right), \bar{I}_\Delta \right) dl \right]_{\{\bar{p},\bar{q},\bar{r},\bar{\sigma},\bar{\delta}\}} = \Phi(t) - \Phi(0); \end{aligned}$$

where the lines above the symbols and designations $\{\bar{p}, \bar{q}, \bar{r}, \bar{\sigma}, \bar{\delta}\}$ indicate the parameters, which belong to the heteroclinic solutions. As the final result of the comparison we can obtain the following values for the heteroclinic parameters:

$$(2.24)\quad \begin{cases} \bar{\delta}(t) = \delta\left(\bar{w}_\delta(t), \bar{l}(t)\right) = \bar{w}_\delta(t) - \bar{v}_\delta(t); \;\Leftrightarrow\; \bar{w}_\delta(t) = \bar{\delta}(t) + \bar{v}_\delta(t); \\ \bar{\delta}(t) = \bar{\omega}_\delta t - \bar{v}_\delta(t) + \delta_0; \quad \bar{w}_\delta(t) = \bar{\omega}_\delta t + w_0; \\ \bar{\omega}_\delta = \sigma_* = \Delta/\bar{C}_1 - (\Delta - EB)/(B - \bar{C}_2); \\ w_0 \overset{df}{=} \delta_0; \;\; \bar{I}_\Delta = \bar{\Delta} = \Delta; \end{cases}$$

It is very important to note the “coincidence” of the “new” Hamiltonian form and the “old” Hamiltonian form after canonical transformation $\tilde{\mathcal{H}} = \mathcal{H}$ (whereas $\partial W/\partial t = 0$), that do not change the substantial structure of the dynamical system; so we can take into our consideration only corresponding “conjunctional” reciprocal transformations between the “old” and “new” canonical variables, if needed. This fact allows us to use the set of canonical variables which can collect a part of “new” and a part of “old” canonical variables. Therefore, we can include into our further investigation the “old” positional canonical pare $\{l, L\}$; and for the description of the rotating phase we will use the both forms, including the “old” Cartesian pare $\{\delta, \Delta\}$ for the natural dynamics modeling, and the “new” action-angle pare $\{w_\delta, I_\Delta\}$ for the Melnikov-Wiggins function evaluating via formal substituting the “conjunctional” heteroclinic solutions.

Conclusion

In this part of the two-part paper the attitude motion of the asymmetrical magnetized DSSC in the constant magnetic field (it corresponds to the equatorial circle orbits of the Earth/planets) at the implementation of the important regime of the cylindrical precession was considered; the corresponding main models and heteroclinc solutions (including the obtained action-angle time-dependencies) are constructed/presented. These models are most complete in the sense of the possible cases of the constructional and mass-inertia asymmetry of the DSSC. Also the Hamiltonian form and required conjunctional expressions were obtained for the considered types of the asymmetry and perturbations. All of these models and solutions will be used in the next part of this paper for the investigation of the chaotic aspects of the DSSC perturbed motion, and for the chaos-suppressing tasks solution.

So, we would like to invite our readers to reading the second part of this paper (The Part II – The heteroclinic chaos investigation).

Acknowledgments

This work is partially supported by the Russian Foundation for Basic Research (RFBR#15-08-05934-A), and by the Ministry education and science of the Russian Federation (in the framework of the design part of the State Assignments to higher education institutions and research organizations in the field of scientific activity – project #9.1004.2014/K).

References

Andoyer H. (1923), Cours de Mecanique Celeste, Paris: Gauthier-Villars.

Anishchenko V.S., Astakhov V.V., Neiman A.B., Vadivasova T.E., Schimansky-Geier L. (2007), Nonlinear Dynamics of Chaotic and Stochastic Systems. Tutorial and Modern Development. Springer-Verlag, Berlin.

Arkhangelskiĭ U.A. (1977), Analytical rigid body dynamics. Nauka, Moscow.

Arnold V.I. (1964), Instability of dynamical systems with several degrees of freedom, Doklady Akademii Nauk SSSR, 156 (1), pp. 9-12.

Aslanov V.S. (2012), Integrable cases in the dynamics of axial gyrostats and adiabatic invariants, Nonlinear Dynamics, Volume 68, Numbers 1-2, pp. 259-273.

Aslanov V.S., Doroshin A.V. (2002), Stabilization of a reentry vehicle by a partial spin-up during uncontrolled descent. Cosmic Research 40 (2), pp. 178-185.

Aslanov V.S., Doroshin A.V. (2010), Chaotic dynamics of an unbalanced gyrostat. Journal of Applied Mathematics and Mechanics, Volume 74, Issue 5, pp.525-535.

Aslanov V.S., Yudintsev V.V. (2012), Dynamics and chaos control of gyrostat satellite, Chaos, Solitons & Fractals 45, pp.1100–1107.

Aslanov V.S., Yudintsev V.V. (2013), Dynamics and control of dual-spin gyrostat spacecraft with changing structure, Celestial Mechanics and Dynamical Astronomy, Volume 115, Issue 1, pp. 91-105.

Aslanov V.S., Yudintsev V.V. (2014), Dynamics and Chaos Control of Asymmetric Gyrostat Satellites, Cosmic Research, Vol. 52, No. 3, pp. 216–228.

Bainum P. M., Fuechsel P. G., Mackison D. L. (1970), Motion and Stability of a Dual-Spin Satellite With Nutation Damping, Journal of Spacecraft and Rockets, Vol. 7, No. 6, pp. 690-696.

Bao-Zeng Y., Jiafang X. (2007), Chaotic attitude maneuvers in spacecraft with a completely liquid-filled cavity, Journal of Sound and Vibration 302, pp. 643–656.

Baozeng Y., Jiafang X. (2007), Chaotic attitude maneuvers in spacecraft with a completely liquid-filled cavity, Journal of Sound and Vibration 302, pp. 643–656.

Bao-Zeng Yue (2008), Heteroclinic bifurcations in completely liquid-filled spacecraft with flexible appendage, Nonlinear Dyn 51:317–327.

Bao-Zeng Yue (2011), Study on the Chaotic Dynamics in Attitude Maneuver of Liquid-Filled Flexible Spacecraft, AIAA Journal, Vol. 49, No. 10, pp. 2090-2099.

Beletskiĭ V.V., Pivovarov M.L., Starostin E.L. (1996), Regular and chaotic motions in applied dynamics of a rigid body. Chaos 6 (2).

Beletsky V.V. (1995), Reguläre und chaotische Bewegung starrer Körper. Teubner.

Boccaletti S., Kurths J., Osipov G., Valladares D.L., Zhou C.S. (2002), The synchronization of chaotic systems. Physics Reports 366,pp. 1–101.

Burov A.A. (2013), Conservative methods of controlling the rotation of a gyrostat, Journal of Applied Mathematics and Mechanics 77, pp.195– 204.

Celletti A., Lhotka C. (2014), Transient times, resonances and drifts of attractors in dissipative rotational dynamics, Commun. Nonlinear Sci. Numer. Simulat. 19, pp. 3399–3411.

Chaikin S.V. (2013), Bifurcations of the relative equilibria of a gyrostat satellite for a special case of the alignment of its gyrostatic moment, Journal of Applied Mathematics and Mechanics 77, pp. 477– 485.

Chen H.-K. (2002), Chaos and chaos synchronization of a symmetric gyro with linear-plus-cubic damping, Journal of Sound and Vibration (2002) 255(4), pp. 719-740.

Chen Li-Qun, Liu Yan-Zhu (2002), Chaotic attitude motion of a magnetic rigid spacecraft and its control, International Journal of Non-Linear Mechanics 37, pp. 493–504.

Chen Li-Qun, Liu Yan-Zhu, Cheng Gong (2002), Chaotic attitude motion of a magnetic rigid spacecraft in a circular orbit near the equatorial plane, Journal of the Franklin Institute 339, pp. 121–128.

Cheng G. and Liu Y. Z. (2000), Chaotic motion of an asymmetric gyrostat in the magnetic field of the earth, Acta Mechanica 141, pp.125-134.

Chinnery A. E., Hall C. D. (1995), Motion of a Rigid Body with an Attached Spring-Mass Damper, J. Guidance Control Dyn. Vol. 18, No. 6, pp.1404-1409.

Cochran, J.E., Shu, P.H., Rew, S.D. (1982), Attitude motion of asymmetric dual-spin spacecraft. J. Guid. Control Dyn. 5, pp.37–42.

Deprit A. (1967), A free rotation of a rigid body studied in the phase plane, American Journal of Physics 35, pp.424-428.

Doroshin A.V. (2008a), Evolution of the Precessional Motion of Unbalanced Gyrostats of Variable Structure. Journal of Applied Mathematics and Mechanics 72, pp. 259–269.

Doroshin A.V. (2008b), Synthesis of Attitude Motion of Variable Mass Coaxial Bodies, WSEAS Transactions on systems and control, Issue 1, Volume 3, pp. 50-61.

Doroshin A.V. (2009), Attitude Control of Spider-type Multiple-rotor Rigid Bodies Systems. Proceedings of the World Congress on Engineering 2009, London, U.K. Vol II, pp.1544-1549.

Doroshin A.V. (2010), Analysis of attitude motion evolutions of variable mass gyrostats and coaxial rigid bodies system, International Journal of Non-Linear Mechanics, Volume 45, Issue 2, pp. 193–205.

Doroshin A.V. (2011), Modeling of chaotic motion of gyrostats in resistant environment on the base of dynamical systems with strange attractors, Communications in Nonlinear Science and Numerical Simulation, Volume 16, Issue 8, pp. 3188–3202.

Doroshin A.V. (2012), Heteroclinic dynamics and attitude motion chaotization of coaxial bodies and dual-spin spacecraft, Communications in Nonlinear Science and Numerical Simulation, Volume 17, Issue 3, pp. 1460–1474.

Doroshin A.V. (2013a), Exact solutions for angular motion of coaxial bodies and attitude dynamics of gyrostat-satellites. International Journal of Non-Linear Mechanics, Volume 50, pp. 68-74.

Doroshin A.V. (2013b), Exact Solutions in Attitude Dynamics of a Magnetic Dual-Spin Spacecraft and a Generalization of the Lagrange Top, WSEAS Transactions on systems, Issue 10, Volume 12, pp. 471-482.

Doroshin A.V. (2014a), Chaos and its avoidance in spinup dynamics of an axial dual-spin spacecraft, Acta Astronautica, Volume 94, Issue 2, pp. 563-576.

Doroshin A.V. (2014b), Homoclinic solutions and motion chaotization in attitude dynamics of a multi-spin spacecraft, Communications in Nonlinear Science and Numerical Simulation, Volume 19, Issue 7, pp. 2528–2552.

Doroshin A.V. (2014c), Multi-spin spacecraft and gyrostats as dynamical systems with multiscroll chaotic attractors, Proceedings of 2014 Science and Information Conference, SAI 2014, pp. 882-887.

Doroshin A.V., Krikunov M.M. (2014), Attitude Dynamics of a Spacecraft with Variable Structure at Presence of Harmonic Perturbations, Appl. Math. Modelling, Volume 38, Issues 7‐8, pp. 2073‐2089.

Doroshin A.V. (2015), Heteroclinic chaos in attitude dynamics of a dual-spin spacecraft at small oscillations of its magnetic moment, WSEAS Transactions on systems. In press.

El-Gohary Awad (2005), Optimal control of damping rotation of a rigid body using fly wheels with friction, Chaos, Solitons and Fractals 24, pp. 207–221.

El-Gohary Awad (2006), New control laws for stabilization of a rigid body motion using rotors system, Mechanics Research Communications, Volume 33, Issue 6, pp. 818-829.

El-Gohary Awad (2009), Chaos and optimal control of steady-state rotation of a satellite-gyrostat on a circular orbit, Chaos, Solitons & Fractals, Volume 42, pp. 2842-2851.

Elipe A., Lanchares V. (2008), Exact solution of a triaxial gyrostat with one rotor, Celestial Mechanics and Dynamical Astronomy, Issue 1-2, Volume 101, pp. 49-68.

Ge Z.-M., Lin T.-N. (2002), Chaos, Chaos Control And Synchronization Of A Gyrostat System, Journal of Sound and Vibration, Volume 251, Issue 3, pp. 519-542.

Ge Z.-M., Lin T.-N. (2003), Chaos, Chaos Control And Synchronization Of Electro-Mechanical Gyrostat System, Journal of Sound and Vibration 259(3), pp. 585–603.

Golubev V.V. (1953), Lectures on Integration of the Equations of Motion of a Rigid Body about a Fixed Point. Moscow: State Publishing House of Theoretical Literature.

Guckenheimer J., Holmes P. (1983), Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields, Springer, Berlin.

Guirao J.L.G., Vera J.A. (2010), Dynamics of a gyrostat on cylindrical and inclined Eulerian equilibria in the three-body problem, Acta Astronautica 66, pp. 595 - 604.

Gutnik S.A., Sarychev V.A. (2014), Dynamics of an axisymmetric gyrostat satellite. Equilibrium positions and their stability, Journal of Applied Mathematics and Mechanics 78, pp. 249–257.

Hall C. D. (1997), Momentum Transfer Dynamics of a Gyrostat with a Discrete Damper, J. Guidance Control Dyn., Vol. 20, No. 6, pp. 1072-1075.

Hall C. D. (1998), Escape from gyrostat trap states, J. Guidance Control Dyn. 21, pp. 421-426.

Hall C.D., Rand R.H. (1994), Spinup Dynamics of Axial Dual-Spin Spacecraft, Journal of Guidance, Control, and Dynamics, Vol. 17, No. 1, pp. 30–37.

Holmes P. J., Marsden J. E. (1983), Horseshoes and Arnold diffusion for Hamiltonian systems on Lie groups, Indiana Univ. Math. J. 32, pp. 273-309.

Holms P. J. (1990), Poincaré, celestial mechanics, dynamical-systems theory and chaos – Physics Reports (Review Section of Physics Letters) 193, No. 3, pp. 137—163 (North-Holland).

Iñarrea M. (2009), Chaos and its control in the pitch motion of an asymmetric magnetic spacecraft in polar elliptic orbit// Chaos, Soliton and Fractals 40, pp. 1637-1652.

Iñarrea M., Lanchares V. (2000), Chaos in the reorientation process of a dual-spin spacecraft with time-dependent moments of inertia, International Journal of Bifurcation and Chaos, Vol. 10, No. 5, pp. 997-1018.

Iñarrea M., Lanchares V., Rothos V. M., Salas J. P. (2003), Chaotic rotations of an asymmetric body with time-dependent moment of inertia and viscous drag, International Journal of Bifurcation and Chaos, Vol. 13, No. 2, pp. 393-409.

Ivin E.A. (1985), Decomposition of variables in task about gyrostat motion, Mathematics and Mechanics series, no.3, Vestnik MGU (Transactions of Moscow's University), pp.69–72.

Kinsey K. J., Mingori D. L., Rand R. H. (1996), Non-linear control of dual-spin spacecraft during despin through precession phase lock, J. Guidance Control Dyn. 19, pp. 60-67.

Kozlov V.V. (1980), Methods of Qualitative Analysis in the Dynamics of a Rigid Body, Gos. Univ., Moscow.

Kuang J.L., Meehan P.A., Leung A.Y.T. (2006a), On the chaotic rotation of a liquid-filled gyrostat via the Melnikov–Holmes–Marsden integral, International Journal of Non-Linear Mechanics 41, pp. 475 – 490.

Kuang J.L., Meehan P.A., Leung A.Y.T. (2006b), Suppressing chaos via Lyapunov–Krasovskii's method, Chaos, Solitons and Fractals 27, pp. 1408–1414.

Kuang Jinlu, Tan Soonhie, Arichandran Kandiah, Leung A.Y.T. (2001), Chaotic dynamics of an asymmetrical gyrostat, International Journal of Non-Linear Mechanics 36, pp. 1213-1233.

Leung A. Y. T., Kuang J. L. (2004), Spatial chaos of 3-D elastica with the Kirchhoff gyrostat analogy using Melnikov integrals, Int. J. Numer. Meth. Eng. 61, pp.1674–1709.

Leung A.Y.T., Kuang J.L. (2007), Chaotic rotations of a liquid-filled solid, Journal of Sound and Vibration 302, pp. 540–563.

Likins P. W. (1986), Spacecraft Attitude Dynamics and Control - A Personal Perspective on Early Developments, J. Guidance Control Dyn. Vol. 9, No. 2, pp. 129-134.

Lin Yiing-Yuh, Wang Chin-Tzuo (2014), Detumbling of a rigid spacecraft via torque wheel assisted gyroscopic motion, Acta Astronautica 93, pp. 1–12.

Meechan P.A., Asokanthan S.F. (2002), Control of chaotic instabilities in a spinning spacecraft with dissipation using Lyapunov's method, Chaos, Solitons and Fractals 13, pp. 1857-1869.

Melnikov V.K. (1963), On the stability of the centre for time-periodic perturbations, Trans. Moscow Math. Soc. No.12, pp. 1-57.

Meng Y., Hao R., Chen Q. (2014), Attitude stability analysis of a dual-spin spacecraft in halo orbits, Acta Astronautica 99, pp. 318–329.

Mingori D. L. (1969), Effects of Energy Dissipation on the Attitude Stability of Dual-Spin Satellites, AIAA Journal, Vol. 7, No. 1, pp. 20-27.

Nazari M., Butcher E.A. (2014), On the stability and bifurcation analysis of dual-spin spacecraft, Acta Astronautica 93, pp.162–175.

Neishtadt A.I., Pivovarov M.L. (2000), Separatrix crossing in the dynamics of a dual-spin satellite. Journal of Applied Mathematics and Mechanics, Volume 64, Issue 5, pp. 709-714.

Or A.C. (1998a), Chaotic Motions of a Dual-Spin Body, Transactions of the ASME, Vol.65, pp. 150-156.

Or A.C. (1998b), Dynamics of an Asymmetric Gyrostat, Journal of Guidance, Control, and Dynamics, Vol.21, No.3, pp. 416-420.

Pecora L.M., Carroll T.L., Johnson G.A., Mar D.J. (1997), Fundamentals of synchronization in chaotic systems, concepts, and applications. Chaos 7 (4), pp. 520-543.

Peng J., Liu Y. (2000), Chaotic motion of a gyrostat with asymmetric rotor, International Journal of Non-Linear Mechanics, Volume 35, Issue 3, pp. 431-437.

Poincaré, H. (1899), Les Methodes Nouvelles de La Mécanique Celeste, Vols. 1-3 (Gauthier Villars, Paris).

Rubanovskii V.M. (1991), Bifurcation and stability of the relative equilibria of a gyrostat satellite, Journal of Applied Mathematics and Mechanics, Volume 55, Issue 4, pp. 450–455.

Sadov Y.A. (1970), The action-angle variables in the Euler-Poinsot problem, Journal of Applied Mathematics and Mechanics, vol. 34, issue 5, pp. 922-925 (PMM “Prikladnaja matematika i mehanika”, vol.34, №5, pp. 962-964).

Sandfry R.A., Hall C.D. (2007), Bifurcations of relative equilibria of an oblate gyrostat with a discrete damper, Nonlinear Dyn 48, pp. 319–329.

Sarychev V.A., Mirer S.A., Isakov A.V. (1981), Dual-spin satellites with gyro-damping, Acta Astronautica, Volume 9, Issue 5, pp. 285–289.

Sarychev V.A., Mirer S. A. (2001), Relative equilibria of a gyrostat satellite with internal angular momentum along a principal axis, Acta Astronautica Vol. 49, No. 11, pp. 641– 644.

Seo In-Ho, Leeghim Henzeh, Bang Hyochoong (2008), Nonlinear momentum transfer control of a gyrostat with a discrete damper using neural networks, Acta Astronautica 62, pp. 357 – 373.

Serret, J. A. (1866), Mémoire sur l’emploi de la méthode de la variation des arbitraires dans la théorie des mouvements de rotation. Mémoires de l’Academie des Sciences de Paris, Vol. 55, pp. 585-616.

Shirazi K.H., Ghaffari-Saadat M.H. (2004), Chaotic motion in a class of asymmetrical Kelvin type gyrostat satellite, International Journal of Non-Linear Mechanics, Volume 39, Issue 5, pp. 785-793.

Tabor M., Chaos and Integrability in Nonlinear Dynamics: An Introduction. Wiley, John & Sons, New York, 1989.

Tong X., Tabarrok B. (1995), Chaotic motion of an asymmetric gyrostat in the gravitational field, International Journal of Non-Linear Mechanics Volume 30, Issue 3, pp.191–203.

Vera J.A. (2013), The gyrostat with a fixed point in a Newtonian force field: Relative equilibria and stability, J. Math. Anal. Appl. 401, pp. 836–849.

Wiggins S. (1988), Global Bifurcations and Chaos : Analytical Methods (Applied mathematical sciences : vol. 73). Springer-Verlag.

Wiggins S., Shaw S.W. (1988), Chaos and Three-Dimentional Horseshoes in Slowly Varying Oscillators, Journal of Applied Mechanics, Vol.55, pp. 959-968.

Wiggins S. (1990), Introduction to applied nonlinear dynamical systems and chaos. Springer-Verlag.

Wiggins S. (1992), Chaotic Transport in Dynamical Systems. Springer-Verlag.

Wittenburg J. (1975), Beitrage zur dynamik von gyrostaten, Accademia Nazional dei Lincei, Quaderno N. 227.

Wittenburg J. (1977), Dynamics of Systems of Rigid Bodies. Stuttgart: Teubner.

Zhou L., Chen Y., Chen F. (2009), Stability and chaos of a damped satellite partially filled with liquid, Acta Astronautica 65, pp. 1628–1638.