Hamiltonian Dynamics of Spider-Type Multirotor Rigid Bodies Systems

Anton V. Doroshin

Theoretical Mechanics Department, Samara State Aerospace University, SSAU, 34, Moskovskoe Shosse str., Samara, 443086, Russia

Abstract. This paper sets out to develop a spider-type multiple-rotor system which can be used for attitude control of spacecraft. The multirotor system contains a large number of rotor-equipped rays, so it was called a “Spider-type System”, also it can be called “Rotary Hedgehog”. These systems allow using spinups and captures of conjugate rotors to perform compound attitude motion of spacecraft. The paper describes a new method of spacecraft attitude reorientation and new mathematical model of motion in Hamilton form. Hamiltonian dynamics of the system is investigated with the help of Andoyer-Deprit canonical variables. These variables allow obtaining exact solution for hetero- and homoclinic orbits in phase space of the system motion, which are very important for qualitative analysis.

Keywords: Spider-Type System, Spacecraft, Reaction Wheels, Conjugate Spinup, Rotor Capture, Andoyer-Deprit Variables, Hetero(Homo)clinic Orbits, Exact Solution.
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INTRODUCTION

Research into attitude motions of rigid body systems has been and remains one of the most important themes of theoretical and applied mechanics. Dynamics of the attitude motion of such systems is a classical mechanical research topic. Basic aspects of this motion were studied by Euler, Lagrange, Kovalevskaya, Zhukovsky, Volterra, Wangerin, Wittenburg. The study of the dynamics of rigid bodies remains very important in modern science and engineering.

Among the basic directions of modern research into the framework of the indicated problem it is possible to highlight the following points: mathematical modeling and analysis of multibody systems motion [1], multibody spacecraft (SC) attitude dynamics and control [2]-[18], multibody systems approach to vehicle dynamics and computer-based techniques [19], simulation of multibody system motion [21], multibody dynamics in computational mechanics [20].

If we speak about practical use of system of rigid bodies dynamics research results we have to note first of all SC with momentum wheels, reaction wheels and control moment gyroscopes (dual-spin satellites, gyrostats, space stations, space telescopes, etc.) [1-18]. This paper sets out to develop a new multiple-rotor system, which also can be used for attitude control of a SC. Due to the large number of rays with rotors, we called the system a “Rotor-type spider”.

1. MECHANICAL MODEL OF THE SYSTEM AND ATTITUDE REORIENTATION METHOD

1.1. Mechanical and Mathematical Models of the System

We shall investigate an attitude motion of multirotor systems about fixed point *O*, as depicted in Fig.1.

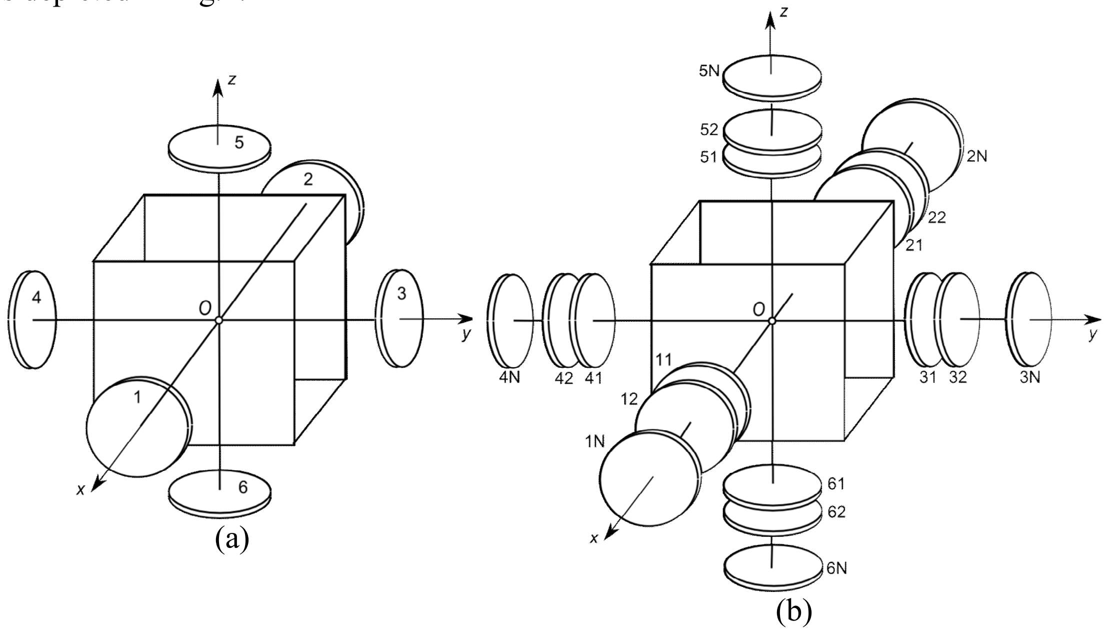


FIGURE 1. Multirotor rigid bodies systems

Firstly we consider rotor-type spider with six rotors which spin about general orthogonal axis of main (central) body (Fig.1-a). Let's assume symmetry of rotors disposition with respect to point *O* and equivalence of their mass-inertia parameters. Angular momentum of the system in projections onto the axes of frame *Oxyz* connected with main body is defined by

$$\mathbf{K} = \mathbf{K}_m + \mathbf{K}_r \tag{1}$$

$$\mathbf{K}_m = \begin{bmatrix} \tilde{A}p \\ \tilde{B}q \\ \tilde{C}r \end{bmatrix} + \left(4J + 2I\right) \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \quad \mathbf{K}_r = I \begin{bmatrix} \sigma_1 + \sigma_2 \\ \sigma_3 + \sigma_4 \\ \sigma_5 + \sigma_6 \end{bmatrix} \tag{2}$$

where $\mathbf{K}_m$ is angular moment of a main rigid body with resting ("frozen") rotors; $\mathbf{K}_r$ is relative angular moment of rotors; $\boldsymbol{\omega}=[p, q, r]^{\mathrm{T}}$ is vector of absolute angular velocity of main body; σ_i is relative angular velocity of *i*-th rotor with respect to main body; $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are general moments of inertia of main body; *I* is longitudinal moments of inertia of single rotor; *J* is equatorial moments of inertia of single rotor calculated about point *O*.

Motion equations of the multirotor system can be obtained considering the change of angular momentum in frame *Oxyz*

$$\frac{d\mathbf{K}}{dt}+\boldsymbol{\omega}\times\mathbf{K}=\mathbf{M}^e \tag{3}$$

where $\mathbf{M}^e$ is principal moment of the external forces. Eq. (3) can be rewritten as

$$\begin{cases} A\dot{p}+I\dot{\sigma}^{12}+\left(C-B\right)qr+I\left(q\sigma^{56}-r\sigma^{34}\right)=M_x^e \\ B\dot{q}+I\dot{\sigma}^{34}+\left(A-C\right)pr+I\left(r\sigma^{12}-p\sigma^{56}\right)=M_y^e \\ C\dot{r}+I\dot{\sigma}^{56}+\left(B-A\right)pq+I\left(p\sigma^{34}-q\sigma^{12}\right)=M_z^e \end{cases} \tag{4}$$

In the last equations following terms are present

$$\begin{gathered} \sigma^{ij}=\sigma_i+\sigma_j, \qquad A=\tilde{A}+4J+2I \\ B=\tilde{B}+4J+2I,\ C=\tilde{C}+4J+2I \end{gathered} \tag{5}$$

We need to add equations of rotors relative motion. These equations can also be written on the base of the law of the change in the angular momentum

$$\begin{cases} I\left(\dot{p}+\dot{\sigma}_1\right)=M_1^i+M_{1x}^e;\ I\left(\dot{p}+\dot{\sigma}_2\right)=M_2^i+M_{2x}^e \\ I\left(\dot{q}+\dot{\sigma}_3\right)=M_3^i+M_{3y}^e;\ I\left(\dot{q}+\dot{\sigma}_4\right)=M_4^i+M_{4y}^e \\ I\left(\dot{r}+\dot{\sigma}_5\right)=M_5^i+M_{5z}^e;\ I\left(\dot{r}+\dot{\sigma}_6\right)=M_6^i+M_{6z}^e \end{cases} \tag{6}$$

where M_j^i is a principal moment of the internal forces acting between main body and j-th rotor; $M_{jx}^e, M_{jy}^e, M_{jz}^e$ are principal moments of external forces acting only at j-th rotor.

Equation systems (4) and (6) together completely describe the attitude dynamics of the rotor-type spider (Fig.1-a).

Motion equations (4) and (6) corresponding to the spider with six rotors can be generalized for description of attitude dynamics of rotor-type spider with 6*N* rotors (Fig.1-b). As presented in Fig.1-b multirotor system has got *N* rotors on every ray - *N* rotor layers (levels). It is assumed that each layer contains equal rotors. Similarly to previous case we can obtain the same equation system (4) for attitude motion of the system with *N* rotor layers (levels), but expressions will contain the following new terms

$$\begin{gathered} \sigma^{ij}=\sum_{l=1}^{N}\left(\sigma_{il}+\sigma_{jl}\right), \qquad A=\tilde{A}+\sum_{l=1}^{N}\left(4J_l+2I_l\right) \\ B=\tilde{B}+\sum_{l=1}^{N}\left(4J_l+2I_l\right),\ C=\tilde{C}+\sum_{l=1}^{N}\left(4J_l+2I_l\right) \\ \mathbf{K}_m=\begin{bmatrix} Ap \\ Bq \\ Cr \end{bmatrix}, \qquad \mathbf{K}_r=\sum_{l=1}^{N}I_l\begin{bmatrix} \sigma_{1l}+\sigma_{2l} \\ \sigma_{3l}+\sigma_{4l} \\ \sigma_{5l}+\sigma_{6l} \end{bmatrix} \end{gathered} \tag{7}$$

where σ_{kl} is the relative angular velocity of the *kl*-th rotor (Fig.1-b) with respect to main body; I_l and J_l are longitudinal and equatorial moments of inertia (calculated about point *O*) of the rotor corresponding to the *l*-th layer.

Equations of rotors' relative motion are given by

$$\begin{cases} I\left(\dot{p}+\dot{\sigma}_{1l}\right)=M_{1l}^{i}+M_{1lx}^{e};\ I\left(\dot{p}+\dot{\sigma}_{2l}\right)=M_{2l}^{i}+M_{2lx}^{e} \\ I\left(\dot{q}+\dot{\sigma}_{3l}\right)=M_{3l}^{i}+M_{3ly}^{e};\ I\left(\dot{q}+\dot{\sigma}_{4l}\right)=M_{4l}^{i}+M_{4ly}^{e} \\ I\left(\dot{r}+\dot{\sigma}_{5l}\right)=M_{5l}^{i}+M_{5lz}^{e};\ I\left(\dot{r}+\dot{\sigma}_{6l}\right)=M_{6l}^{i}+M_{6lz}^{e} \end{cases} \tag{8}$$

where $l=1..N$ and therefore we have got N systems like (8) for each kl-th rotor.

Equation system (4) with terms (7) and N systems like (8) completely describe of attitude dynamics of the rotor-type spider with $6N$ rotors (Fig.1-b).

Thus, we have dynamic equations of attitude motion. Let's define kinematic parameters and corresponding kinematic equations. We will use well-known [23] Euler parameters $\{\lambda_0,\lambda_1,\lambda_2,\lambda_3\}$ describing a finite rotation of main body by an angle χ about an arbitrary unit vector $\mathbf{e}=\left[\cos\alpha,\cos\beta,\cos\gamma\right]^T$ in inertial fixed frame $OXYZ$ which coincides with the initial position of $Oxyz$ (Fig.2).

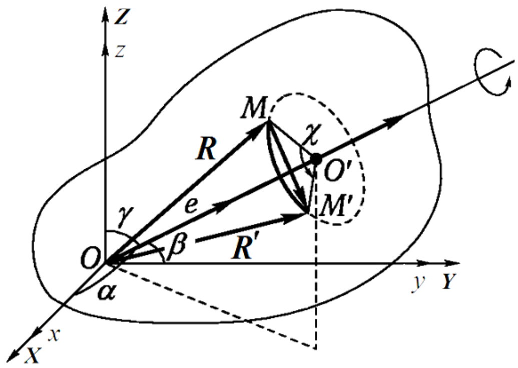


FIGURE 2. Finite rotation

The Euler parameters are defined by

$$\begin{cases} \lambda_0=\cos\dfrac{\chi}{2}; \qquad \lambda_1=\cos\alpha\sin\dfrac{\chi}{2} \\ \lambda_2=\cos\beta\sin\dfrac{\chi}{2};\ \lambda_3=\cos\gamma\sin\dfrac{\chi}{2} \end{cases} \tag{9}$$

Following system of kinematical equation arises for Euler parameters

$$2\dot{\boldsymbol{\lambda}}=\boldsymbol{\Theta}\cdot\boldsymbol{\lambda} \tag{10}$$

where

$$\boldsymbol{\lambda}=\begin{bmatrix}\lambda_0\\ \lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix},\quad \boldsymbol{\Theta}=\begin{bmatrix}0 & -p & -q & -r\\ p & 0 & r & -q\\ q & -r & 0 & p\\ r & q & -p & 0\end{bmatrix} \tag{11}$$

The above set of dynamic and kinematic equations completely describe of attitude motion of multirotor systems.

1.2. The Method of Attitude Reorientation of the System

We give a series of definitions.

Def.1. Conjugate rotors are paired rotors located in the same layer on the opposite rays. For example, rotor 3N and rotor 4N (Fig.1-b) are conjugate rotors (also rotor 12 and rotor 22, etc.).

Def.2. Conjugate spinup mean a process of spinning up conjugate rotors in opposite directions up to a desired value of relative angular velocity with the help of internal moments from main body. Velocities of conjugate rotors will be equal in absolute value and opposite in sign.

Def.3. Rotor capture is an immediate deceleration of rotor relative angular velocity with the help of internal moment from the main body. So, rotor capture means an "instantaneous freezing" of rotor with respect to the main body. The capture can be performed with the help of gear meshing, friction clutch or other methods.

Now we provide an explanation of the attitude reorientation method.

Let's consider conjugate spinup of conjugate rotors 1 and 2 (Fig.1-a) in the absence of external moments $\left(M_x^e = M_y^e = M_z^e = 0\right)$ assuming initial rest of main body and all rotors and mass-inertia symmetry of the system

$$p(0) = q(0) = r(0) = 0; \ \ \forall i: \ \sigma_i(0) = 0 \tag{12}$$

$$A = B = C = D \tag{13}$$

In the simplest case we can use the following piecewise constant of the spinup internal moments

$$M_1^i = \begin{cases} M_{12}, & if \ \ t \in \left[0, t_{12}^s\right], \\ 0, & otherwise; \end{cases} \qquad M_2^i = \begin{cases} -M_{12}, & if \ t \in \left[0, t_{12}^s\right], \\ 0, & otherwise. \end{cases} \tag{14}$$

where t_{12}^s is the time instant of spinup termination of rotors 1 and 2; $M_{12} = \mathrm{const} > 0$.

After the conjugate spinup rotors 1 and 2 will reach an absolute value $S_{12} = M_{12} \cdot t_{12}^s / I$ of relative angular velocity $(\sigma_1 = S_{12}, \ \sigma_2 = -S_{12})$ but the main body will remain in rest; angular momentum of system will remain equal to zero. After the conjugate spinup we capture rotor 1 at time instant t_1^c $\left(t_1^c > t_{12}^s\right)$. Then the relative angular velocity of rotor 1 becomes null $(\sigma_1' = 0)$, but main body will take absolute angular velocity p and rotor 2 will change relative angular velocity up to σ_2'. Conservation of angular momentum of full system makes it possible to write

$$Ap + I\sigma_2' = 0 \tag{15}$$

Similarly, the conservation of angular momentum of rotor 2 makes it possible to write

$$I\left(p + \sigma_2'\right) = -IS_{12} \tag{16}$$

Numerical values for angular velocities after caption of rotor 1 are obtained from expressions (15) and (16)

$$p = \frac{IS_{12}}{A - I}; \quad \sigma_2' = -\frac{AS_{12}}{A - I} \tag{17}$$

At the time t_2^c $\left(t_2^c > t_1^c\right)$ we capture rotor 2 and then all bodies (main body and both conjugate rotors) return to absolute rest.

Thus we can conclude that conjugate spinup and two serial captures of conjugate rotors result in a piecewise constant angular velocity of main body

$$p = \begin{cases} 0, & t \in \left[0, t_1^c\right) \cup \left(t_2^c, \infty\right) \\ P = \dfrac{IS_{12}}{A - I}, & t \in \left[t_1^c, t_2^c\right] \end{cases} \tag{18}$$

It can be used for main body angular reorientation about corresponding axis. In our case the main body performed the rotation about Ox axis by a finite angle

$$\varphi_x = \frac{IS_{12}\left(t_2^c - t_1^c\right)}{A - I} \tag{19}$$

2. HAMILTONIAN DYNAMICS OF THE SYSTEM

2.1. Hamiltonian Form of Motion Equations

We consider spider-system with N layers of rotors (Fig.1-b). Kinetic energies of six rotors in j-layer are

$$\begin{aligned} &2T_{1j} = J_j\left(q^2 + r^2\right) + I_j\left(p + \sigma_{1j}\right)^2; \quad 2T_{2j} = J_j\left(q^2 + r^2\right) + I_j\left(p + \sigma_{2j}\right)^2 \\ &2T_{3j} = J_j\left(p^2 + r^2\right) + I_j\left(q + \sigma_{3j}\right)^2; \quad 2T_{4j} = J_j\left(p^2 + r^2\right) + I_j\left(q + \sigma_{4j}\right)^2 \\ &2T_{5j} = J_j\left(p^2 + q^2\right) + I_j\left(r + \sigma_{5j}\right)^2; \quad 2T_{6j} = J_j\left(p^2 + q^2\right) + I_j\left(r + \sigma_{6j}\right)^2 \end{aligned} \tag{20}$$

Kinetic energy of the system has the following expression

$$\begin{aligned} &T = T_0 + \sum_{j=1}^{N} T_j; \quad 2T_0 = \tilde{A}p^2 + \tilde{B}q^2 + \tilde{C}r^2; \quad T_j = \sum_{i=1}^{6} T_{ij}; \\ &2T_j = \left(2I_j + 4J_j\right)\left(p^2 + q^2 + r^2\right) + \\ &+ 2I_j\left(p\left[\sigma_{1j} + \sigma_{2j}\right] + q\left[\sigma_{3j} + \sigma_{4j}\right] + r\left[\sigma_{5j} + \sigma_{6j}\right]\right) + I_j \sum_{i=1}^{6} \sigma_{ij}^2 \end{aligned} \tag{21}$$

Let us describe the system motion with the help of Hamiltonian formalism and Andoyer-Deprit canonical variables [24-26]. In this approach the main body attitude motion is described by three angles φ_3, φ_2 and l, which correspond to rotation about axes OZ, about angular moment direction and about Oz, correspondingly (Fig.3). Expressions for canonical momentums (conjugate momenta) have following appearances

$$L = \frac{\partial \mathrm{T}}{\partial \dot{l}} = \mathbf{K} \cdot \mathbf{k}, \quad G = \frac{\partial \mathrm{T}}{\partial \dot{\varphi}_2} = \mathbf{K} \cdot \mathbf{s} = K, \quad H = \frac{\partial \mathrm{T}}{\partial \dot{\varphi}_3} = \mathbf{K} \cdot \mathbf{k}', \quad \Delta_{ij} = \frac{\partial \mathrm{T}}{\partial \dot{\delta}_{ij}} = \frac{\partial \mathrm{T}}{\partial \sigma_{ij}} \tag{22}$$

δ_{ij} is angle of relative rotation of ij-rotor with respect to the main body $\left(\dot{\delta}_{ij} = \sigma_{ij}\right)$.

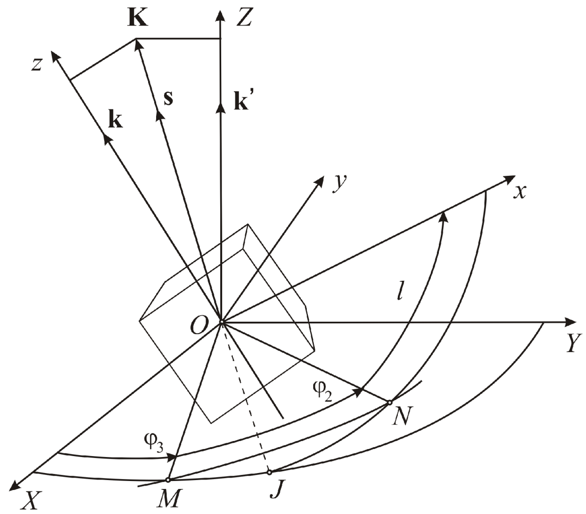


FIGURE 3. Andoyer-Deprit variables and main body coordinate-frame

Canonical momentums L, H are components of angular momentum of the system onto axes *Oz* and *OZ*, and *G* is equal to value of the angular momentum. It is important to note, that the following relation holds

$$\frac{L}{G} = \cos\theta \tag{23}$$

where θ is the nutation angle, which describes the attitude orientation of the main body with respect to the direction of the vector of the system angular momentum.

Components of system angular momentum can be expressed with the help of Andoyer-Deprit canonical variables, and, also, can be written with the help of expressions (7):

$$\begin{aligned}
K_x &= \sqrt{G^2 - L^2}\sin l = Ap + \sum_{j=1}^{N} I_j\left(\sigma_{1j} + \sigma_{2j}\right) \\
K_y &= \sqrt{G^2 - L^2}\cos l = Bq + \sum_{j=1}^{N} I_j\left(\sigma_{3j} + \sigma_{4j}\right) \\
K_z &= L = Cr + \sum_{j=1}^{N} I_j\left(\sigma_{5j} + \sigma_{6j}\right), \qquad \left(L \le G\right)
\end{aligned} \tag{24}$$

Expressions for canonical momentum of relative motion of j-layer's rotors are

$$\begin{aligned}
\Delta_{1j} &= I_j\left(p + \sigma_{1j}\right); \quad \Delta_{2j} = I_j\left(p + \sigma_{2j}\right) \\
\Delta_{3j} &= I_j\left(q + \sigma_{3j}\right); \quad \Delta_{4j} = I_j\left(q + \sigma_{4j}\right) \\
\Delta_{5j} &= I_j\left(r + \sigma_{5j}\right); \quad \Delta_{6j} = I_j\left(r + \sigma_{6j}\right)
\end{aligned} \tag{25}$$

From (25) it follows

$$\sum_{j=1}^{N} I_j\left(\sigma_{1j}+\sigma_{2j}\right)=\sum_{j=1}^{N}\left(\Delta_{1j}+\Delta_{2j}\right)-2p\sum_{j=1}^{N} I_j \tag{26}$$

$$\sum_{j=1}^{N} I_j\left(\sigma_{3j}+\sigma_{4j}\right)=\sum_{j=1}^{N}\left(\Delta_{3j}+\Delta_{4j}\right)-2q\sum_{j=1}^{N} I_j \tag{27}$$

$$\sum_{j=1}^{N} I_j\left(\sigma_{5j}+\sigma_{6j}\right)=\sum_{j=1}^{N}\left(\Delta_{5j}+\Delta_{6j}\right)-2r\sum_{j=1}^{N} I_j \tag{28}$$

Components of angular velocity of the main body can be written with the help of Eq. (26)-(28) and (25) as follows

$$p=\frac{1}{\hat{A}}\left[\sqrt{G^2-L^2}\sin l-\sum_{j=1}^{N}\left(\Delta_{1j}+\Delta_{2j}\right)\right] \tag{29}$$

$$q=\frac{1}{\hat{B}}\left[\sqrt{G^2-L^2}\cos l-\sum_{j=1}^{N}\left(\Delta_{3j}+\Delta_{4j}\right)\right] \tag{30}$$

$$r=\frac{1}{\hat{C}}\left[L-\sum_{j=1}^{N}\left(\Delta_{5j}+\Delta_{6j}\right)\right] \tag{31}$$

where $\hat{A}=A-2\sum_{j=1}^{N} I_j$; $\hat{B}=B-2\sum_{j=1}^{N} I_j$; $\hat{C}=C-2\sum_{j=1}^{N} I_j$.

Taking into account Eq. (29)-(31), we can express kinetic energy (21) in Andoyer-Deprit variables:

$$\begin{aligned}2T=&\left(G^2-L^2\right)\left[\frac{\sin^2 l}{\hat{A}}+\frac{\cos^2 l}{\hat{B}}\right]+\frac{1}{\hat{C}}\left(L-\sum_{j=1}^{N}\left[\Delta_{5j}+\Delta_{6j}\right]\right)^2-\\&-2\sqrt{G^2-L^2}\left\{\frac{\sin l}{\hat{A}}\cdot\sum_{j=1}^{N}\left[\Delta_{1j}+\Delta_{2j}\right]+\frac{\cos l}{\hat{B}}\cdot\sum_{j=1}^{N}\left[\Delta_{3j}+\Delta_{4j}\right]\right\}+\\&+\frac{1}{\hat{A}}\left(\sum_{j=1}^{N}\left[\Delta_{1j}+\Delta_{2j}\right]\right)^2+\frac{1}{\hat{B}}\left(\sum_{j=1}^{N}\left[\Delta_{3j}+\Delta_{4j}\right]\right)^2+\sum_{j=1}^{N}\sum_{i=1}^{6}\frac{\Delta_{ji}^2}{I_j}\end{aligned} \tag{32}$$

Assuming the absence of internal and external potential forces, the Hamiltonian of the system takes on form

$$\begin{aligned}H=T=&\frac{G^2-L^2}{2}\left[\frac{\sin^2 l}{\hat{A}}+\frac{\cos^2 l}{\hat{B}}\right]+\frac{1}{2\hat{C}}\left(L-D_{56}\right)^2-\\&-\sqrt{G^2-L^2}\left\{\frac{D_{12}\sin l}{\hat{A}}+\frac{D_{34}\cos l}{\hat{B}}\right\}+\frac{D_{12}^2}{2\hat{A}}+\frac{D_{34}^2}{2\hat{B}}+T_R=h=\text{const},\end{aligned} \tag{33}$$

where

$$
\begin{aligned}
&D_{12} = \sum_{j=1}^{N} \left[\Delta_{1j} + \Delta_{2j} \right], \quad D_{34} = \sum_{j=1}^{N} \left[\Delta_{3j} + \Delta_{4j} \right] \\
&D_{56} = \sum_{j=1}^{N} \left[\Delta_{5j} + \Delta_{6j} \right], \quad T_R = \frac{1}{2} \sum_{j=1}^{N} \sum_{i=1}^{6} \frac{\Delta_{ji}^2}{I_j}.
\end{aligned} \tag{34}
$$

We will use the following convenient constant for energy level description

$$
\tilde{h} = h - \left(\frac{D_{12}^2}{2\hat{A}} + \frac{D_{34}^2}{2\hat{B}} \right) - T_R \tag{35}
$$

Now we can write the canonical equations for Andoyer-Deprit variables

$$
\begin{aligned}
&\dot{Q} = \frac{\partial H}{\partial P}; \quad \dot{P} = -\frac{\partial H}{\partial Q} \\
&Q = \left\{ l, \varphi_2, \varphi_3, \delta_{ij} \right\}, \ P = \left\{ L, G, H, \Delta_{ij} \right\}
\end{aligned} \tag{36}
$$

It is seen that the Hamiltonian (33) depends on coordinate l only, and therefore only canonical momentum L is variable, other momentum are constant

$$
\begin{aligned}
&-\frac{\partial H}{\partial \varphi_2} = -\frac{\partial H}{\partial \varphi_3} = -\frac{\partial H}{\partial \delta_{ij}} = 0, \\
&G = \text{const}; \quad H = \text{const}; \\
&\forall i, j: \quad \Delta_{ij} = \text{const}; \quad D_{ij} = \text{const}.
\end{aligned} \tag{37}
$$

Thus the phase space of the system is completely described by two variables $\{l, L\}$ and the corresponding equations

$$
\begin{aligned}
&\dot{L} = -\frac{\partial H}{\partial l} = -\frac{G^2 - L^2}{2} \left[\frac{1}{\hat{A}} - \frac{1}{\hat{B}} \right] \sin 2l - \sqrt{G^2 - L^2}\, D \sin(l - s) \\
&\dot{l} = \frac{\partial H}{\partial L} = L \left[\frac{1}{\hat{C}} - \left(\frac{1}{2\hat{A}} + \frac{1}{2\hat{B}} \right) - \left(\frac{1}{2\hat{B}} - \frac{1}{2\hat{A}} \right) \cos 2l + \frac{D \cos(l - s)}{\sqrt{G^2 - L^2}} \right] - \frac{D_{56}}{\hat{C}}
\end{aligned} \tag{38}
$$

where

$$
D = \sqrt{\frac{D_{12}^2}{\hat{A}^2} + \frac{D_{34}^2}{\hat{B}^2}}, \quad \cos(s) = \frac{D_{34}}{\hat{B} D} \tag{39}
$$

2.2. The System Phase Portraits, Homo- and Heteroclinic Orbits

Traditionally phase space analysis of the system includes phase portraits (PP) plotting, description of PP structure and critical points bifurcations investigation. Let us carry out a numerical investigation of PP structure and obtain exact solutions for homo- and heteroclinic phase-trajectories (PT) which separate areas of PP.

Figures (Fig.4 and Fig.5) illustrate general cases of PP. Clearly, various PP-types are possible.

2.2.1. Case Of Dynamically Symmetrical Main Body

Firstly let us consider a case of motion of the system with dynamically symmetrical main body $\left(\hat{A}=\hat{B}\right)$. In this case Eq. (38) becomes

$$\begin{aligned} \dot{L} &= -\frac{\partial H}{\partial l} = -D\sqrt{G^2-L^2}\sin\left(l-s\right) \\ \dot{l} &= \frac{\partial H}{\partial L} = L\left[\frac{1}{\hat{C}}-\frac{1}{\hat{A}}+\frac{D\cos\left(l-s\right)}{\sqrt{G^2-L^2}}\right]-\frac{D_{56}}{\hat{C}} \end{aligned} \tag{40}$$

With the help of equation for Hamiltonian (33) phase trajectories can be expressed:

$$\begin{aligned} &\cos\left(l-s\right)=f\left(L\right); \qquad l\left(L\right)=\pm\arccos f\left(L\right)+s \\ &f\left(L\right)=-\frac{1}{D\sqrt{G^2-L^2}}\left[\tilde{h}-\frac{G^2-L^2}{2\hat{A}}-\frac{1}{2\hat{C}}\left(L-D_{56}\right)^2\right] \\ &\tilde{h}=h-\frac{D_{12}^2+D_{34}^2}{2\hat{A}}-T_R; \qquad s=\arccos\frac{D_{34}}{\sqrt{D_{12}^2+D_{34}^2}} \end{aligned} \tag{41}$$

Value s in this case shifts the PP along l-axis under the conservation of PP energy level – it can be shown in Fig.4-c and Fig.4-d. Therefore we can change variable $l\leftarrow\left(l-s\right)$ and assume that s=0.

Take into account (41) and the trigonometric identity, first Eq.(40) can be rewritten as

$$\dot{L}=\mp D\sqrt{G^2-L^2}\sqrt{1-f^2\left(L\right)} \tag{42}$$

Let us obtain exact solutions for heteroclinic separatix PT corresponding to saddle (hyperbolic) points S_{I} (Fig.4). Coordinate numerical values $\left(l=2\pi n,\ L=L_I\right)$ of S_{I} can be easy found as coordinates of critical (fixed) point of the system (40) $\left\{\dot{l}=0,\ \dot{L}=0\right\}$. Energy level of the separatix PT equals the following constant

$$\tilde{h}_I=\frac{G^2-L_I^2}{2\hat{A}}+\frac{\left(L_I-D_{56}\right)^2}{2\hat{C}}-D\sqrt{G^2-L_I^2} \tag{43}$$

Substitution of (43) into Eq. (42) gives the equation for the separatix PT time-dependence $\bar{L}_I\left(t\right)$

$$\dot{\bar{L}}_I=\pm\sqrt{f_1\left(\bar{L}_I\right)f_2\left(\bar{L}_I\right)}, \tag{44}$$

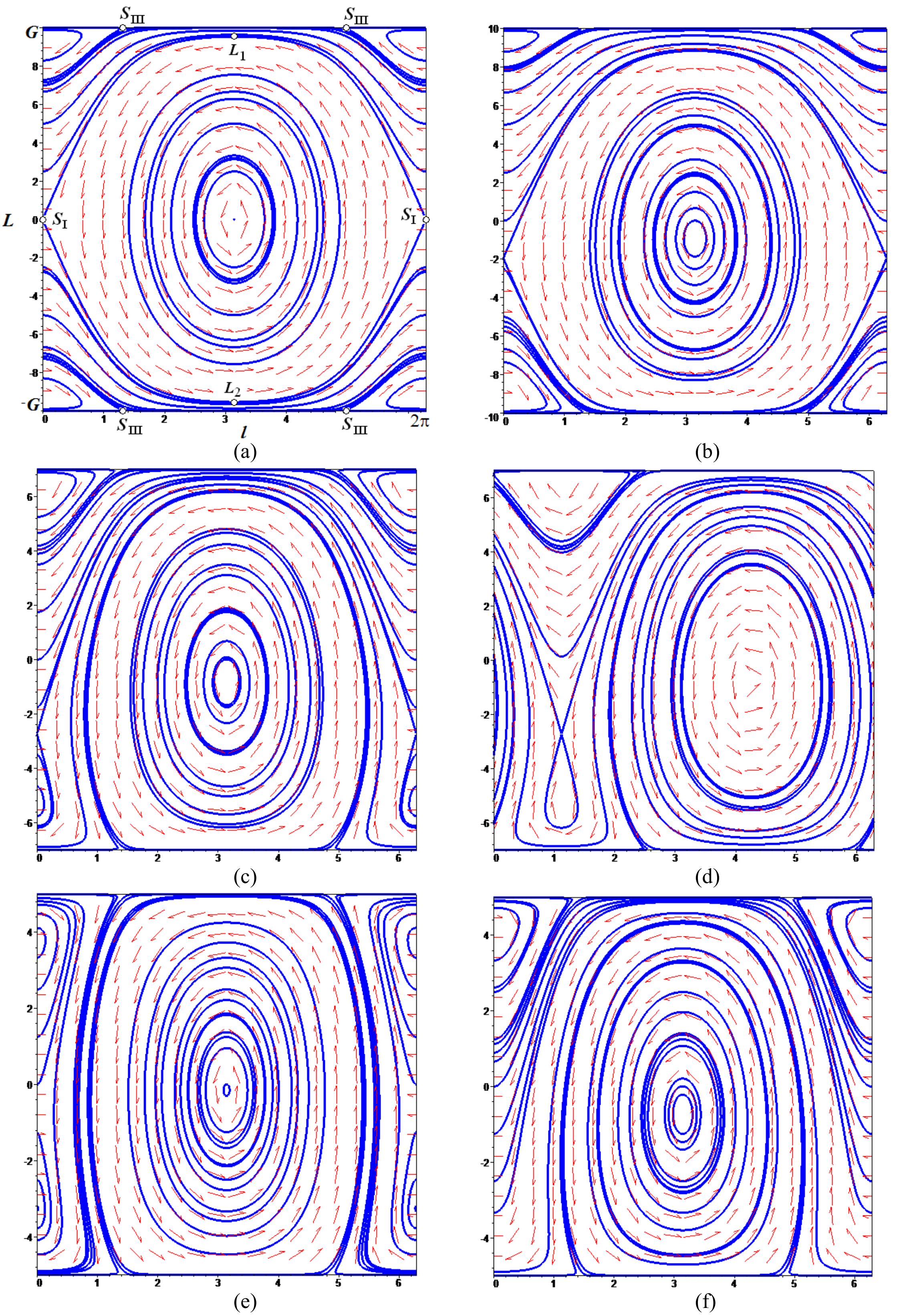
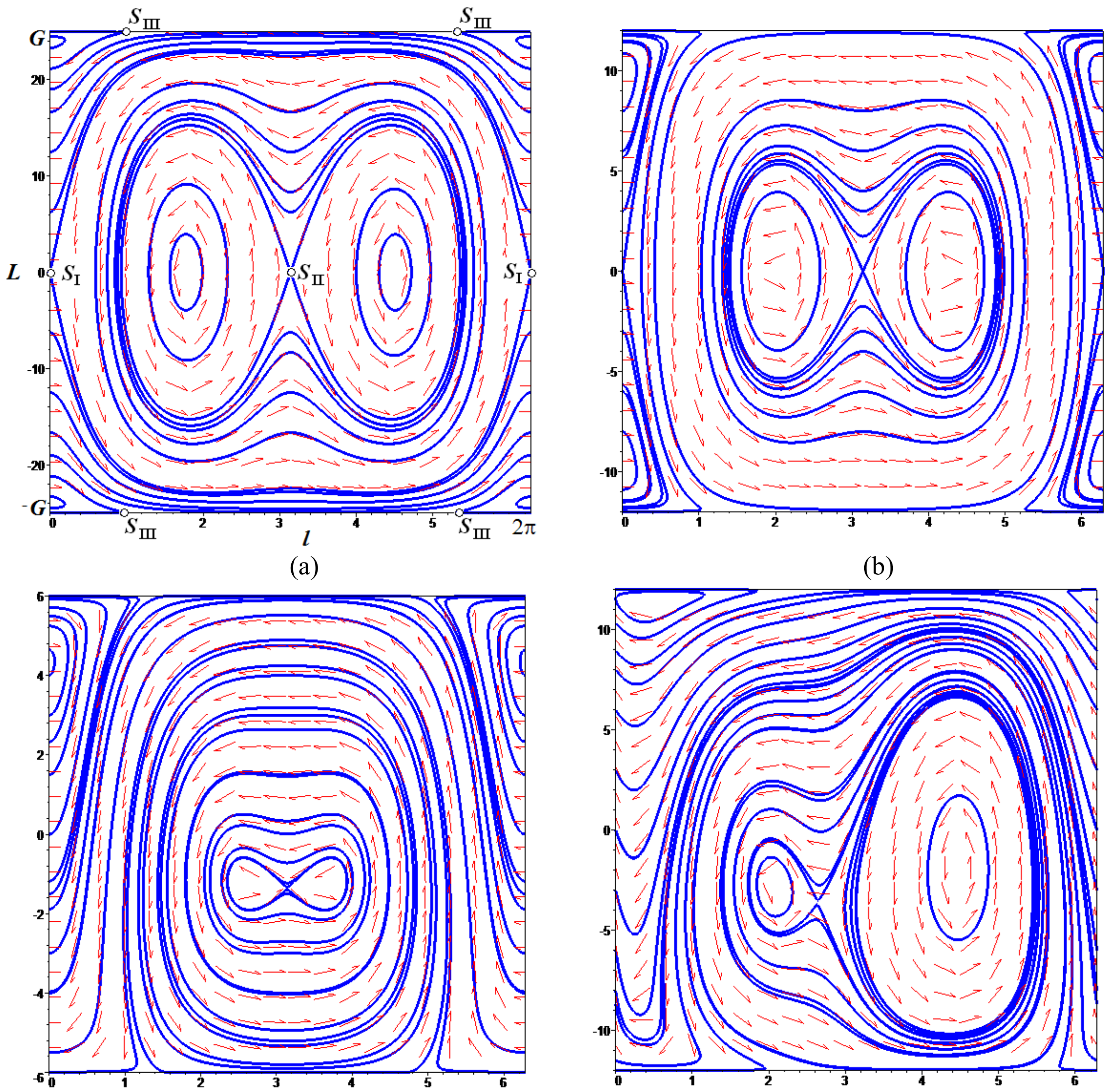


FIGURE 4. Phase portraits of the system.
Normalized moments of inertia of main body: $\hat{A} = \hat{B} = 0.5,\ \hat{C} = 0.7$.

TABLE 1. Calculation parameters for Fig.4.

Momentum	Case (a)	(b)	(c)	(d)	(e)	(f)
G, N·m·s	10	10	7	7	5	5
D_{12}, N·m·s	0	0	0	0.9	0	0
D_{34}, N·m·s	1	1	1	0.44	1	1
D_{56}, N·m·s	0	0.5	0.5	0.5	0.1	0.5



(a) (b) (c) (d)

FIGURE 5. Phase portraits of the system.
Normalized moments of main body: $\hat{A} = 0.5,\ \hat{B} = 0.6,\ \hat{C} = 0.7$.

TABLE 2. Calculation parameters for Fig.5.

Momentum	Case (a)	(b)	(c)	(d)
G, N·m·s	25	12	6	12
D_{12}, N·m·s	0	0	0	0.6
D_{34}, N·m·s	1	1	1	0.7
D_{56}, N·m·s	0	0	0.5	0.9

where

$$f_1\left(\overline{L}_I\right) = -D\sqrt{G^2 - \overline{L}_I^{\,2}} - \left[\tilde{h}_I - \frac{G^2 - \overline{L}_I^2}{2\hat{A}} - \frac{1}{2\hat{C}}\left(\overline{L}_I - D_{56}\right)^2\right],$$
$$f_2\left(\overline{L}_I\right) = -D\sqrt{G^2 - \overline{L}_I^{\,2}} + \left[\tilde{h}_I - \frac{G^2 - \overline{L}_I^2}{2\hat{A}} - \frac{1}{2\hat{C}}\left(\overline{L}_I - D_{56}\right)^2\right].$$

Quadratic polynomial $f_1\left(\overline{L}_I\right)$ has multiple root L_I corresponding to S_1 point value. Quadratic polynomial $f_2\left(\overline{L}_I\right)$ has two different roots corresponding to L_1 and L_2 points. Eq. (44) in this case becomes

$$\dot{\overline{L}}_I = \pm\left(\overline{L}_I - L_I\right)\sqrt{k\left(\overline{L}_I - L_1\right)\left(\overline{L}_I - L_2\right)}, \tag{45}$$

where $k = \left(\dfrac{1}{2\hat{A}} - \dfrac{1}{2\hat{C}}\right)^2$.

Change of variable $x = \overline{L}_I - L_I$ gives the following final quadrature for Eq. (45)

$$\begin{aligned} &\pm t\sqrt{k} = \int \frac{dx}{x\sqrt{R(x)}} = \frac{1}{\sqrt{a_1}} \ln \frac{2a_0 + a_1 x + 2\sqrt{a_0 R(x)}}{x} + C \\ &R(x) = a_0 + a_1 x + a_2 x^2, \quad C = \text{const} \end{aligned} \tag{46}$$

From (46) with the help of variable back-substitution we can obtain exact explicit time-dependence $\overline{L}_I\left(t\right)$ of separatix PT in terms of the exponential functions of time.

It is important to note, that similarly we can obtain analytical expression for time-dependence $\overline{L}_{III}\left(t\right)$ corresponding to heteroclinic orbits for S_{III} hyperbolic points (Fig.4).

2.2.2. Case Of Triaxial Main Body

Assume that $\hat{A} < \hat{B} < \hat{C}$ and $D_{12} = D_{56} = 0$. In this case s=0, and PP corresponds to Fig.5-a, Fig.5-b. PP has three types of hyperbolic points: S_{I}, S_{II}, S_{III}. In considered phase space $\{l,\ L\}$ point S_{I} has coordinate values $l = 2\pi n,\ L = 0$, and point S_{II} : $l = \pi + 2\pi n,\ L = 0.$

Energy level of S_{I} is

$$\tilde{h}_I = \frac{G^2}{2\hat{B}} - \frac{GD_{34}}{\hat{B}} \tag{47}$$

Hamiltonian, corresponding to energy level of S_{I} heteroclinic orbit $\left\{\overline{l}_I\left(t\right), \overline{L}_I\left(t\right)\right\}$, becomes

$$H = \frac{G^2 - \overline{L}_I^2}{2}\left[\frac{1}{\hat{A}} + \left(\frac{1}{\hat{B}} - \frac{1}{\hat{A}}\right)\cos^2 \overline{l}_I\right] + \frac{\overline{L}_I^2}{2\hat{C}} - \frac{D_{34}}{\hat{B}}\sqrt{G^2 - \overline{L}_I^2}\cos \overline{l}_I = \tilde{h}_I \tag{48}$$

Taking into account (47) we can express $\cos \overline{l}_I$ from Hamiltonian (48) as solution of quadratic equation

$$\cos \overline{l}_I = \frac{-b \pm \sqrt{Discr}}{2a} \tag{49}$$

$$Discr = b^2 - 4ac = \left(G^2 - \overline{L}_I^2\right)\left(\kappa^2 - \chi^2 \overline{L}_I^2\right);$$

$$a = \frac{G^2 - \overline{L}_I^2}{2}\left(\frac{1}{\hat{B}} - \frac{1}{\hat{A}}\right); \quad b = -\frac{D_{34}}{\hat{B}}\sqrt{G^2 - \overline{L}_I^2}; \quad c = \frac{G^2 - \overline{L}_I^2}{2\hat{A}} + \frac{\overline{L}_I^2}{2\hat{C}} - \frac{G^2}{2\hat{B}} + \frac{GD_{34}}{\hat{B}};$$

$$\kappa^2 = \left[\left(\frac{1}{\hat{A}} - \frac{1}{\hat{B}}\right)G + \frac{D_{34}}{\hat{B}}\right]^2; \quad \chi^2 = \left(\frac{1}{\hat{A}} - \frac{1}{\hat{B}}\right)\left(\frac{1}{\hat{A}} - \frac{1}{\hat{C}}\right).$$

Equation for heteroclinic separatrix orbit $\overline{L}_I(t)$ can be written on the base of (38)

$$\dot{\overline{L}}_I = \sin \overline{l}_I \sqrt{G^2 - \overline{L}_I^2}\left[\sqrt{G^2 - \overline{L}_I^2}\left(\frac{1}{\hat{B}} - \frac{1}{\hat{A}}\right)\cos \overline{l}_I - \frac{D_{34}}{\hat{B}}\right] \tag{50}$$

After substitution of (49) Eq. (50) becomes

$$\dot{\overline{L}}_I = \pm\sqrt{\left(G^2 - \overline{L}_I^2\right) - \frac{\hat{A}^2\hat{B}^2}{\left(\hat{A} - \hat{B}\right)^2}\left(\frac{D_{34}}{\hat{B}} \pm \sqrt{\left(\kappa - \chi \overline{L}_I\right)\left(\kappa + \chi \overline{L}_I\right)}\right)^2}\sqrt{\left(\kappa - \chi \overline{L}_I\right)\left(\kappa + \chi \overline{L}_I\right)} \tag{51}$$

Change of variable

$$S = \sqrt{\left(\kappa - \chi \overline{L}_I\right)\left(\kappa + \chi \overline{L}_I\right)} \tag{52}$$

gives the following differential equation

$$\frac{dS}{\sqrt{\left(\kappa - S\right)\left(\kappa + S\right)\left(\alpha S^2 \mp \beta S + \gamma\right)}} = \pm \chi dt; \tag{53}$$

where

$$\alpha = \frac{1}{\chi^2} - \frac{\hat{A}^2\hat{B}^2}{\left(\hat{A} - \hat{B}\right)^2}; \quad \beta = \frac{2D_{34}\hat{A}^2\hat{B}^2}{\hat{B}\left(\hat{A} - \hat{B}\right)^2}; \quad \gamma = G^2 - \frac{\kappa^2}{\chi^2} - \frac{D_{34}^2\hat{A}^2\hat{B}^2}{\hat{B}^2\left(\hat{A} - \hat{B}\right)^2}.$$

It can be shown that one of roots of quadratic polynomial in (53) has the value $\pm\kappa$. Therefore, Eq. (53) becomes

$$\frac{dS}{\sqrt{\alpha(\kappa-S)(\kappa+S)(S-S_2)(S\pm\kappa)}}=\pm\chi dt \tag{54}$$

where S_2 is the second root of quadratic polynomial in (53). Eq.(54) can be rewritten in two variants depend on the choice of sign in the radical

$$\int\frac{dS}{(\kappa+S)\sqrt{\alpha(\kappa-S)(S-S_2)}}=\pm\int\chi dt;\quad \int\frac{dS}{(\kappa-S)\sqrt{\alpha(\kappa+S)(S_2-S)}}=\pm\int\chi dt \tag{55}$$

Both variants of expressions (55) can be reduced to the final quadrature (46).

In the case of separatrix corresponding to hyperbolic points S_{II} (Fig.5-a) we can conduct an analogous calculation and obtain separatrix homoclinic orbit $\bar{L}_{II}(t)$. The computation again leads to the final quadrature (46), but constants change theirs values:

$$\tilde{h}_{II}=\frac{G^2}{2\hat{B}}+\frac{GD_{34}}{\hat{B}};\qquad \kappa^2=\left[\frac{D_{34}}{\hat{B}}-\left(\frac{1}{\hat{A}}-\frac{1}{\hat{B}}\right)G\right]^2$$

Thus we have obtained analytical solutions in the form of a final quadrature corresponding to homo- and heteroclinic orbits of the system. These exact solutions are important in dynamic analysis of the system. Homo- and heteroclinic orbits time-dependencies are the main element of local investigation of chaotic behavior near the separatrix with the help of Melnicov theory, including Wiggins, Holms, Marsdens results [27-30].

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